Path-Complete Lyapunov Functions

The comparison problem

Hi

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Warsaw

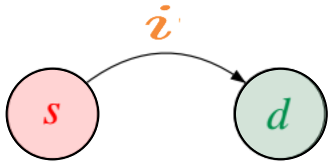
Path-complete Lyapunov functions *Introduction*

Given a discrete switched dynamical system, a **Path-Complete Lyapunov Function** (PCLF) is a pair $(G, \mathcal{V})$ where

- the **graph** $G = (S, E)$ is *Path-Complete* (aka *universal*), and
- we assign a candidate Lyapunov function to each node of the graph, and each of them has to belong to the **template** $\mathcal{V}$ of functions,

such that the following Lyapunov inequalities are satisfied

$$\forall\,(s, d, i) \in E,\ \forall x \in \mathbb{R}^n : V_d(f_i(x)) \leq V_s(x)$$



Lyapunov inequalities

Two parameters with which we can play

a combinatorial component
the graph

an algebraic component
the template

Path-complete Lyapunov functions

Quadratic Lyapunov functions: the queen of stability analysis

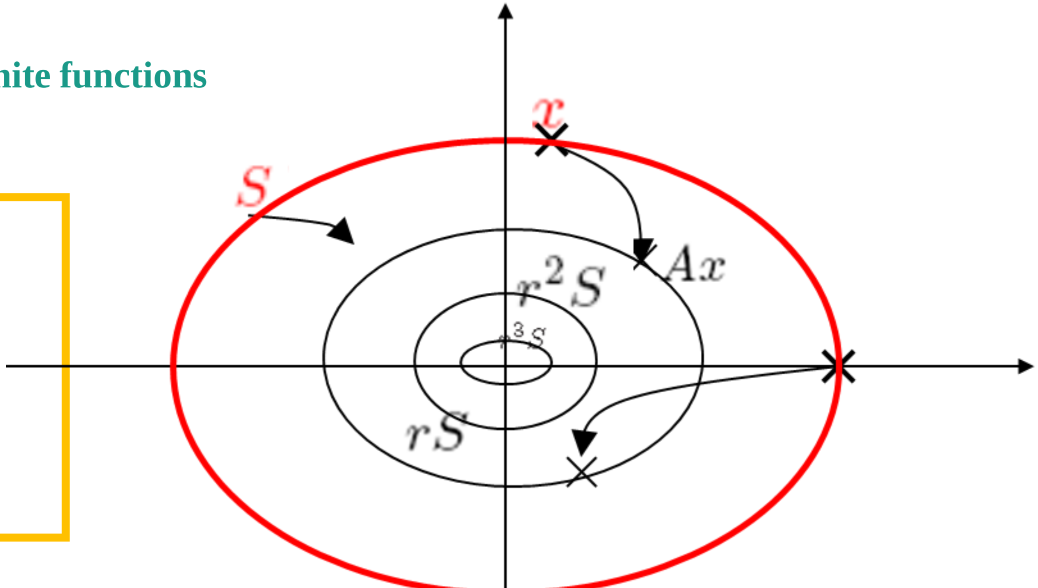
Given a **linear** discrete-time switched dynamical system defined by

$$x(k+1) = A_{\sigma(k)}(x(k))$$

.

One can use the **template** of **quadratic positive definite functions**

$$\begin{array}{llll} \inf_{r\in\mathbb{R}^+} & & r & \\ \text{s.t.} & & & \\ A^T P A & \preceq & r^2 P, & \forall A \in \Sigma \\ P & \succeq & 0. & \end{array}$$



Path-complete Lyapunov functions

Copositive linear norms: another commonly used template

Given a **linear positive** discrete-time switched dynamical system defined by

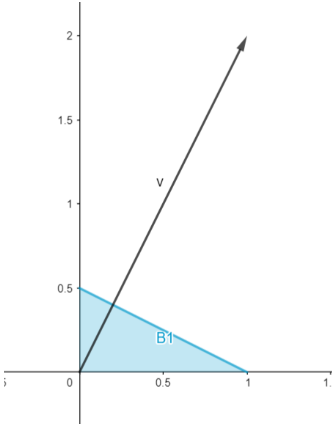
$$x(k+1) \;=\; A_{\sigma(k)}\,(x(k))$$

where the **matrices** $\mathcal{A} := \{A_1, \ldots, A_M\} \subseteq \mathbb{R}^n_{\geq 0}$.

One can use

the **template** $\mathcal{P}$ of **primal linear copositive norms** induced by $v \in \mathbb{R}^n_{>0}$, i.e.

$$\|x\|_v \;:=\; v^\top x, \quad x \in \mathbb{R}^n_{\geq 0}.$$



Several **other templates** are commonly used, such **as Sum-Of-Squares, dual linear copositive norms, or polytopic/zonotopic templates**

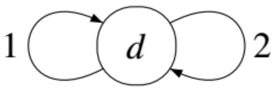
Path-complete Lyapunov functions *Example*

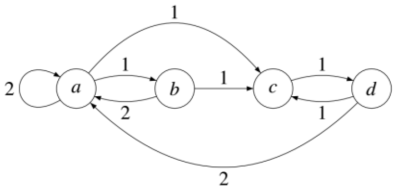
Example

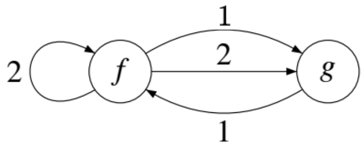
In the **linear** case, we can estimate the **decay rate**, defined as the infimum $\gamma \geq 0$ for which the scaled system $\mathcal{A}_\gamma := \{A_1/\gamma, A_2/\gamma\}$ is stable.

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{30} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{3} \end{bmatrix}$$

➔ γ provides an **index of performance** of my stability criterion: the smaller it is, the best is my criterion







Template \ Graph	Graph 1	Graph 2	Graph 3
Quadratic (SDP)	0.913	0.913	0.910
Linear copositive (LP)	1.000	1.000	0.903

Presentation outline

1. Introduction
2. **The comparison problem**
3. Template-dependent lifts
4. characterization theorems
5. Numerical example
6. Two open problems
7. Conclusion and future work

2.Path-complete Lyapunov functions

Our problem today: the comparison between two graphs

Theorem

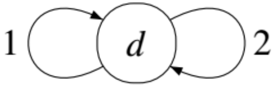
Given two path-complete graphs $\mathcal{G}_1 := (S_1, E_1)$ and $\mathcal{G}_2 := (S_2, E_2)$. The following statements are equivalent :

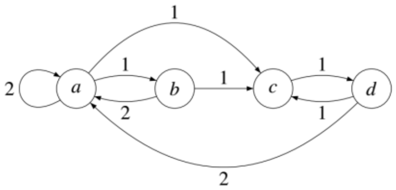
A. $\mathcal{G}_1$ simulates $\mathcal{G}_2$.

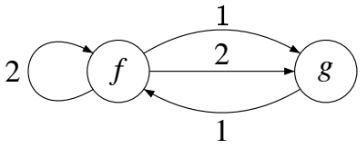
B. $\mathcal{G}_1 \leq \mathcal{G}_2$.

[M. Philippe & R.M. Jungers, *A complete characterization of the ordering of path-complete methods*, 2019]

Definition: $\mathcal{G}_1 \leq \mathcal{G}_2$ if **for any switched system**, and **for any template**, Graph $\mathcal{G}_1$ performs less well than graph $\mathcal{G}_2$. (that is, the upper bound is larger)







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2.Path-complete Lyapunov functions *Comparison*

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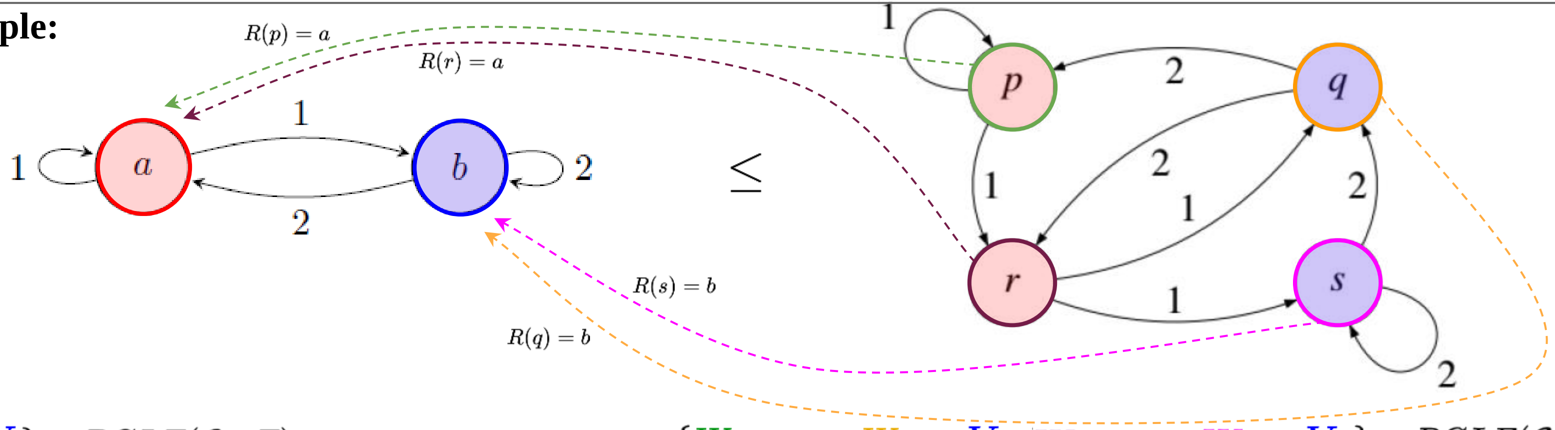
Example:



$$\underbrace{\{V_a, V_b\}}_{\subseteq \mathcal{V}} \in PCLF(\mathcal{G}_1, F) \xrightarrow[\forall F = \{f_1, f_2\}]{} \underbrace{\{W_p := V_a \,, W_q := V_b \,, W_r := V_a \,, W_s := V_b\}}_{\subseteq \mathcal{V}} \in PCLF(\mathcal{G}_2, F)$$

Eureka?

Eureka?

Not quite...

2.Path-complete Lyapunov functions *Comparison*

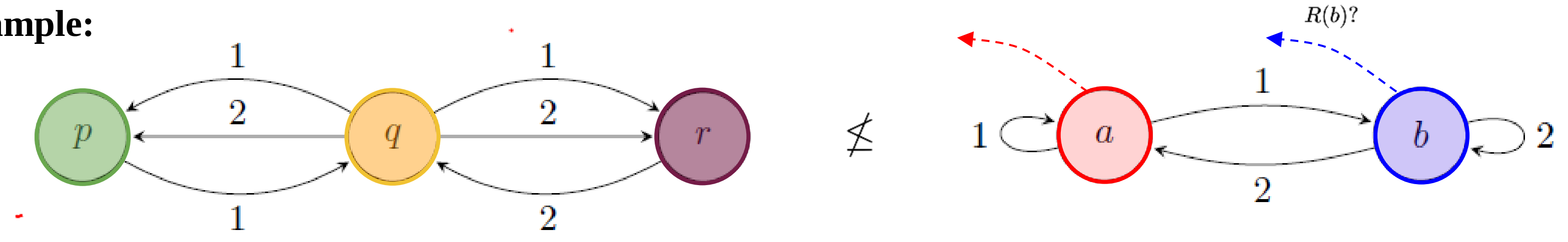
Theorem

Given two path-complete graphs $\mathcal{G}_1 := (S_1, E_1)$ and $\mathcal{G}_2 := (S_2, E_2)$. The following statements are equivalent :

A. $\mathcal{G}_1$ simulates $\mathcal{G}_2$.

B. $\mathcal{G}_1 \leq \mathcal{G}_2$.

Example:



$$\{V_p, V_q, V_r\} \in PCLF(\mathcal{G}_1, F) \xrightarrow[\forall F = \{f_1, f_2\}]{?} \left\{ W_a := \begin{matrix} V_p \\ \text{or} \\ V_q \\ \text{or} \\ V_r \end{matrix} \, , \, W_b := \begin{matrix} V_p \\ \text{or} \\ V_q \\ \text{or} \\ V_r \end{matrix} \right\} \in PCLF(\mathcal{G}_2, F)$$

2. Path-complete Lyapunov functions *Comparison*

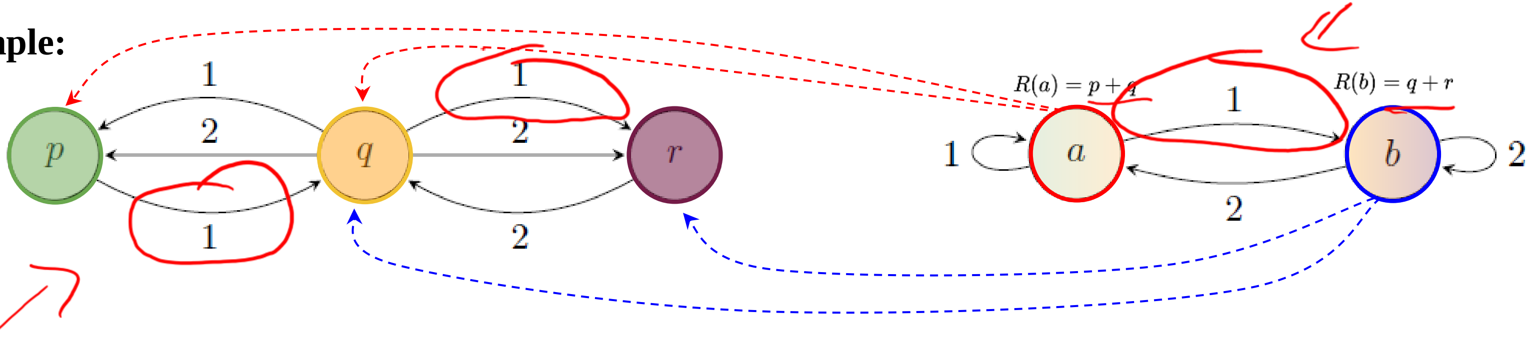
$V_q(A_1 x) \le V_r(x)$ $\qquad$ $V_q + V_r(A_1 x) \le V_r + V_q(x)$

Consider the LMIs described by the following graphs, together with the **quadratic template**
One can build a solution for the right graph by **taking sums of functions of the left graph**

$+$ $V_r(A_1 x) \le V_q(x)$

➔ **If the left graph has a solution, the right graph has one too**

Example:

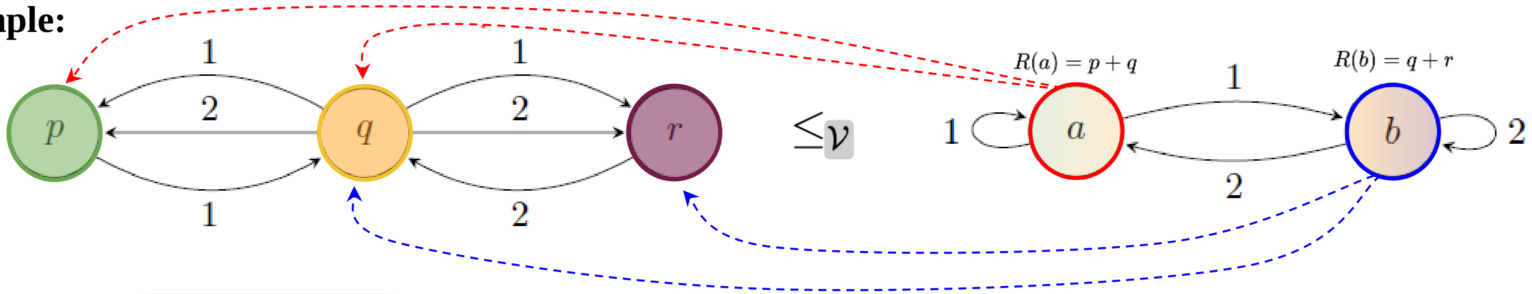


$$\{V_p, V_q, V_r\} \in PCLF(\mathcal{G}_1, F) \xrightarrow[\forall F = \{f_1, f_2\}]{} \{W_a := V_p + V_q, W_b := V_q + V_r\} \in PCLF(\mathcal{G}_2, F)$$

2. Path-complete Lyapunov functions *Comparison*

Even though the right graph is **not intrinsically better than the left one**, it is better **when the template is closed under sum**!

Example:



For any template $\mathcal{V}$ closed under sum,

$$\underbrace{\{V_p, V_q, V_r\}}_{\subseteq \mathcal{V}} \in PCLF(\mathcal{G}_1, F) \xrightarrow[\forall F = \{f_1, f_2\}]{} \underbrace{\{W_a := V_p + V_q,\ W_b := V_q + V_r\}}_{\subseteq \mathcal{V}} \in PCLF(\mathcal{G}_2, F)$$

Presentation outline

1. Introduction
2. The comparison problem
3. **Template-dependent lifts**
4. characterization theorems
5. Numerical example
6. Two open problems
7. Conclusion and future work

3. Template-dependent lifts

Example

$W_{\{a\}} := V_a$

$W_{\{b\}} := V_b$

$W_{\{a,b\}} := \min\{V_a, V_b\} \in \mathcal{V}$

Consider a template $\mathcal{V}$ closed under **min** :

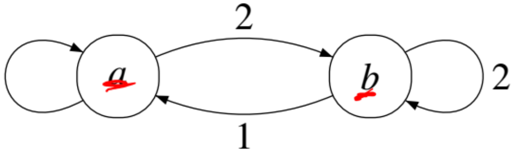
$W_{\{a,b\}}$ 2 1

$x \in E_a$

$\to A_i x \in E_a \cup E_b$

$x \in E_b$

$A_i x \in E_a \cup E_b$



$\mathcal{G} = (S, E)$

One may assume the existence of a **more sophisticated graph**, taking into account **the implicitely existing functions**

3. Template-dependent lifts

Example

Consider a template $\mathcal{V}$ closed under **min** :

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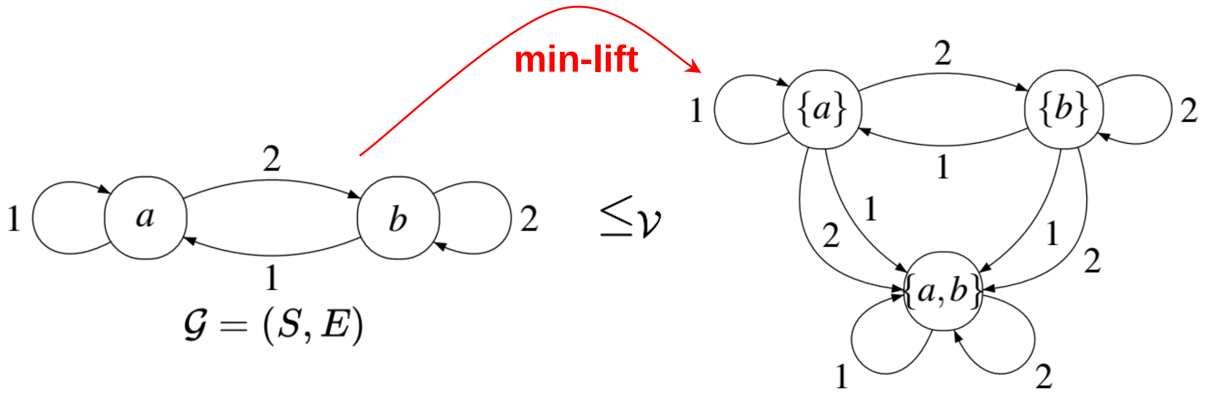

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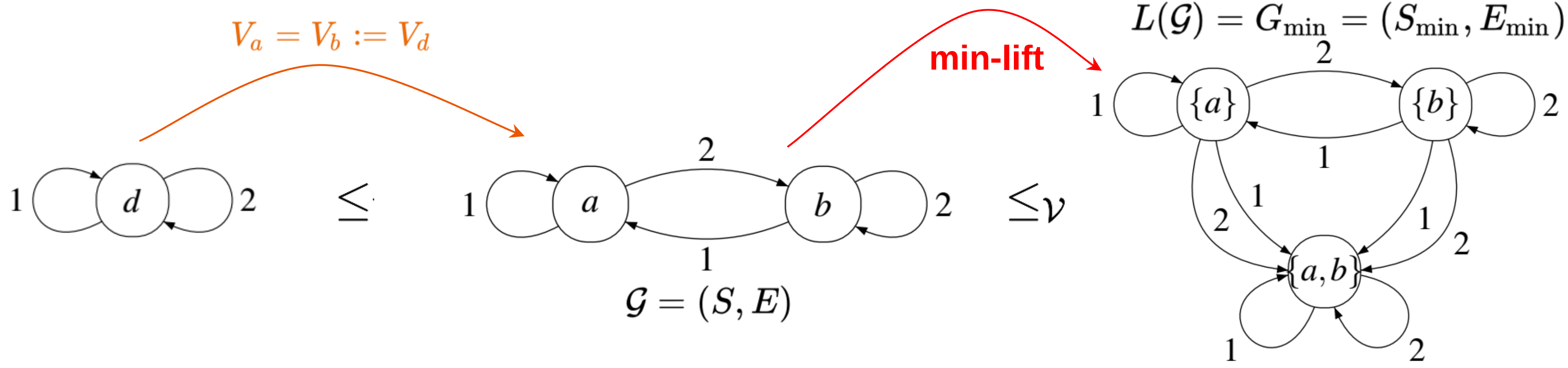

$$V_d(f_1(x)) \leq V_d(x)$$
$$V_d(f_2(x)) \leq V_d(x)$$

$$V_a(f_1(x)) \leq V_a(x)$$
$$V_b(f_2(x)) \leq V_a(x)$$
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$$V_a(f_1(x)) \leq V_b(x)$$

$$W_{\{a\}}(f_1(x)) \leq W_{\{a\}}(x) \quad W_{\{a,b\}}(f_1(x)) \leq W_{\{a\}}(x)$$
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3. Template-dependent lifts

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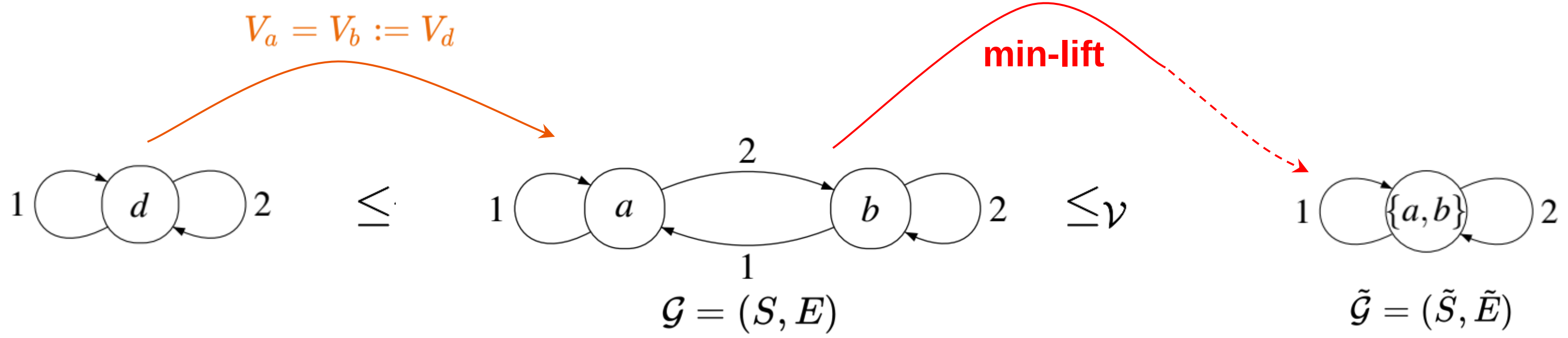

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3. Template-dependent lifts

Example

Consider a template $\mathcal{V}$ closed under **min** :

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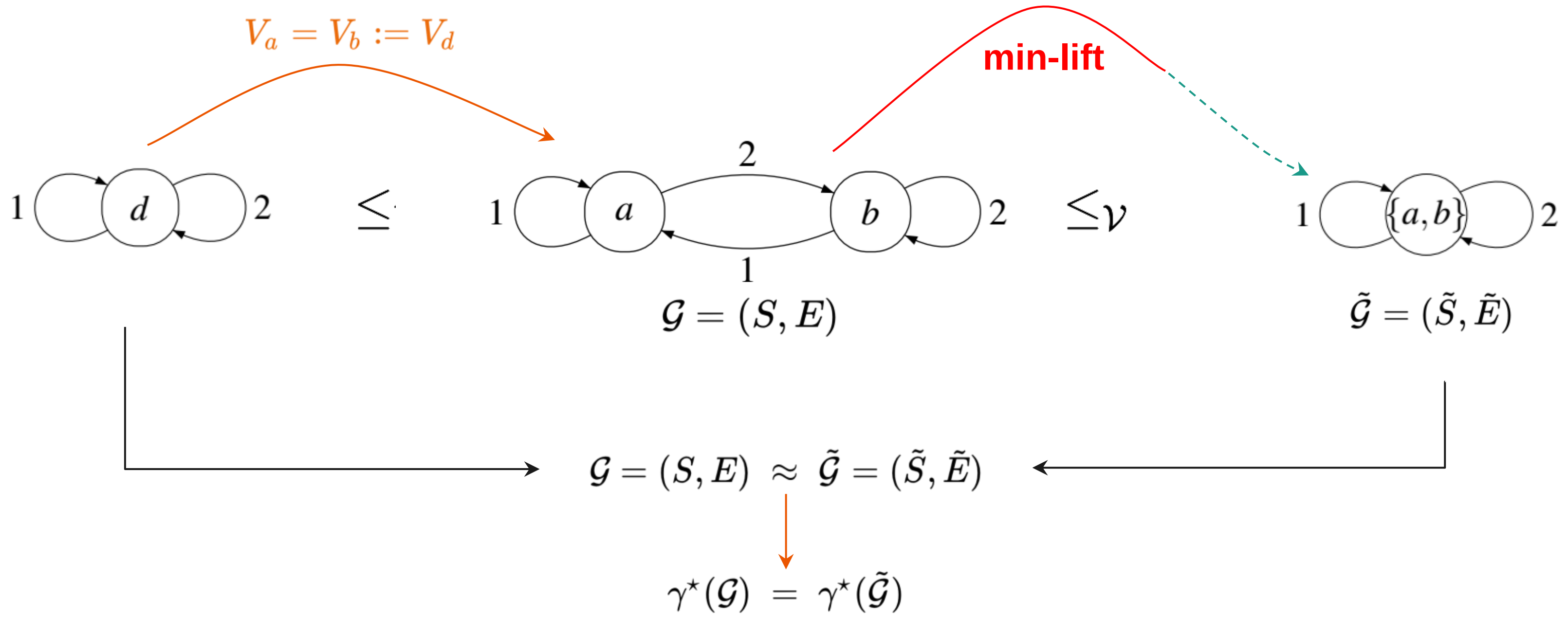


3. Template-dependent lifts *Min lift*

We say that $L : Graphs_M \to Graphs_M$ is a **valid lift with respect to a template** $\mathcal{V}$ if

(1) $\mathcal{G}$ path-complete implies $L(\mathcal{G})$ is path-complete,
(2) $\mathcal{G} \leq_{\mathcal{V}} L(\mathcal{G})$, for all path-complete graph $\mathcal{G}$.

Given a graph $\mathcal{G} = (S, E)$ on $\langle M \rangle$, the **min-lift** is a graph $\mathcal{G}_{\min} = (S_{\min}, E_{\min})$ defined as follows:

(1) The set of nodes $S_{\min}$ is defined by

$$S_{\min} := \{S' \subset S \mid S' \neq \emptyset\} \longrightarrow W_{S'} := \min_{s \in S'} V_s$$

(1) $(A, B, i) \in E_{\min}$ if and only if

$$\forall a \in A, \exists b \in B \text{ s.t. } (a, b, i) \in E \quad \text{(SC)}$$

$$\{V_s \mid s \in S\} \in PCLF(\mathcal{G}, F) \Rightarrow \{W_A := \min_{a \in A} V_a(x) \mid A \in S_{\min}\} \in PCLF(\mathcal{G}_{\min}, F)$$

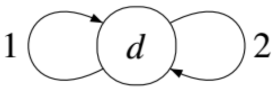
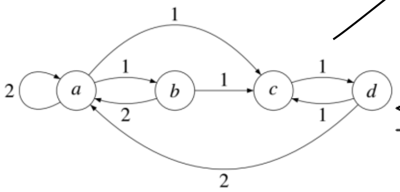
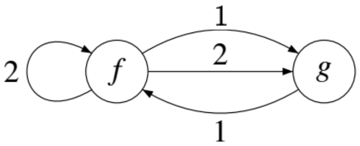
3. Template-dependent lifts *Application of the min-lift*

We consider the linear positive switched system on $M = 2$ modes defined by

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{30} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

We estimate the decay rate.

strongly connected component of the **min-lift**

Template \ Graph	 Graph 1	 Graph 2 $\leq_{\mathcal{P}}$	 Graph 3
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3. Template-dependent lifts *Definition*

We say that $L : Graphs_M \to Graphs_M$ is a **valid lift with respect to a template** $\mathcal{V}$ if

(1) $\mathcal{G}$ path-complete implies $L(\mathcal{G})$ is path-complete,
(2) $\mathcal{G} \leq_{\mathcal{V}} L(\mathcal{G})$, for all path-complete graph $\mathcal{G}$.

→ **Min lift :** $\mathcal{G} \to L(\mathcal{G}) := \mathcal{G}_{\min}$

$W \subseteq \{U_P := \min_{p \in P} V_p \mid P \subseteq S, P \neq \emptyset\} \subseteq \mathcal{V}$ valid for templates closed under **pointwise minimum**

→ **Max lift :** $\mathcal{G} \to L(\mathcal{G}) := \mathcal{G}_{\max}$

$W \subseteq \{U_P := \max_{p \in P} V_p \mid P \subseteq S, P \neq \emptyset\} \subseteq \mathcal{V}$ valid for templates closed under **pointwise maximum**

→ **Sum lift :** $\mathcal{G} \to L(\mathcal{G}) := \mathcal{G}^{\oplus} = \cup_{T \in \mathbb{N}} \mathcal{G}^{\oplus T}$

$W \subseteq \{U_P := \sum_{p \in P} V_p \mid P \subseteq S, P \neq \emptyset\} \subseteq \mathcal{V}$ valid for templates closed under **addition**

Presentation outline

1. Introduction
2. The comparison problem
3. Template-dependent lifts
4. **characterization theorems**
5. Numerical example
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$$G_1 \preceq_{min} G_1^{min}$$

$$G_1^{min} \sim_{im} G_2$$

$$\Downarrow$$

$$G_1^{min} \preceq G_2$$

$$\Downarrow$$

$$G_1 \preceq_{min} G_2$$

4.characterization theorems *Min lift*

Theorem

Given two path-complete graphs $\mathcal{G}_1 := (S_1, E_1)$ and $\mathcal{G}_2 := (S_2, E_2)$. The following statements are equivalent :

1) $\mathcal{G}_{1\,\mathrm{min}}$ simulates $\mathcal{G}_2$.
2) $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$ for any template $\mathcal{V}$ closed under pointwise minimum.

4.characterization theorems *Min lift*

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In practice :

$\mathcal{G}_1 = (S_1, E_1)$

$\mathcal{G}_2 = (S_2, E_2)$

Ouput
Yes or No

Step 1
Computation of the **min** lift

Step 2
Check of a simulation relation

$\mathcal{G}_{1\,\mathrm{min}} = (S_{1\,\mathrm{min}}, E_{1\,\mathrm{min}})$

4. characterization theorems *Sum lift, min lift, and max lift*

Theorem

Given two path-complete graphs $\mathcal{G}_1 := (S_1, E_1)$ and $\mathcal{G}_2 := (S_2, E_2)$. The following statements are equivalent :

1) $\mathcal{G}_1^{\oplus}$ simulates $\mathcal{G}_2$.
2) $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$ for any template $\mathcal{V}$ closed under addition.

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Given two path-complete graphs $\mathcal{G}_1 := (S_1, E_1)$ and $\mathcal{G}_2 := (S_2, E_2)$. The following statements are equivalent :

1) $\mathcal{G}_{1\,\mathrm{max}}$ simulates $\mathcal{G}_2$.
2) $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$ for any template $\mathcal{V}$ closed under pointwise maximum.

Presentation outline

5. Numerical example *Approximation of the JSR*

Given a **linear switched system** with two modes

$$A_1 = \begin{bmatrix} 0.9 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0.6 & 0.9 \\ 0.6 & 0.3 \end{bmatrix}$$

We want to approximate the **joint spectral radius** (JSR) of $\mathcal{A} := \{A_1, A_2\}$, i.e. the infimum value $\gamma \geq 0$ such that the scaled system

$$\mathcal{A}_\gamma := \{A_1/\gamma, A_2/\gamma\}$$

is stable.

5. Numerical example *Approximation of the JSR*

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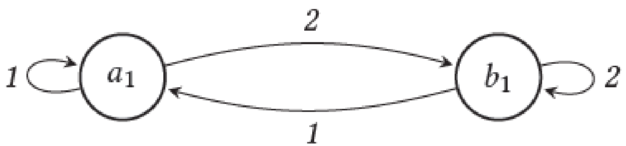
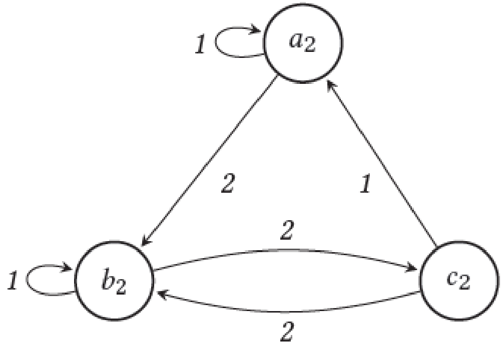


$\mathcal{G}_1 := (S_1, E_1)$



$\mathcal{G}_2 := (S_2, E_2)$

5. Numerical example *Approximation of the JSR*

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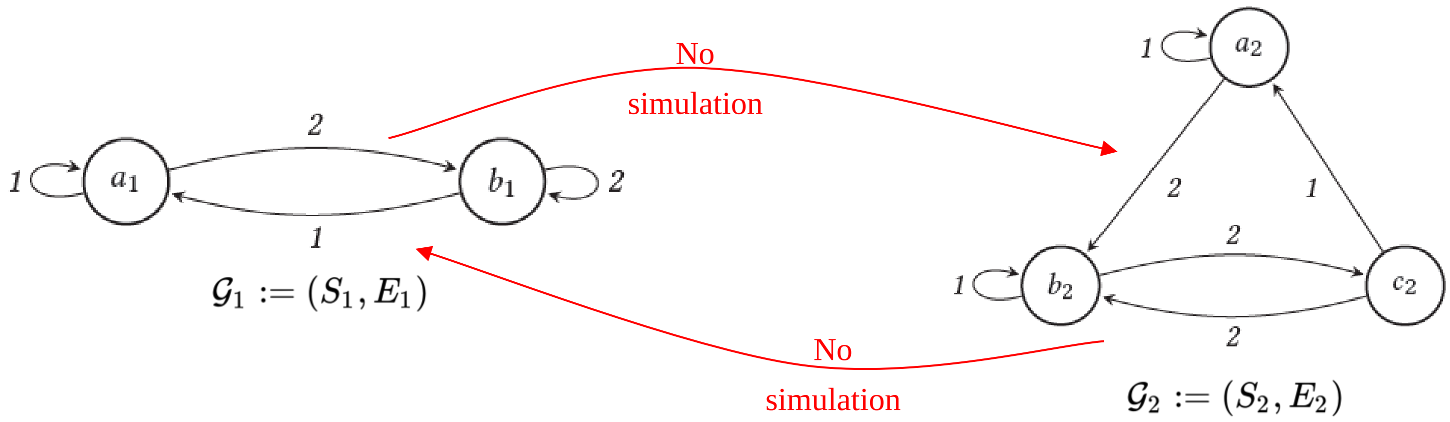

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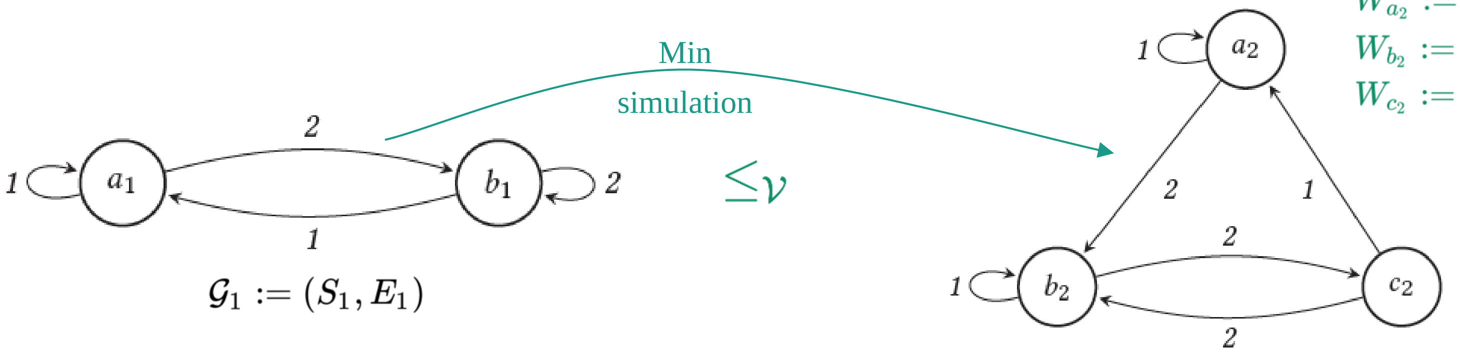
We want to approximate the **joint spectral radius** (JSR) of $\mathcal{A} := \{A_1, A_2\}$.

$$W_{a_2} := V_{a_1}$$
$$W_{b_2} := \min\{V_{a_1}, V_{b_1}\}$$
$$W_{c_2} := V_{b_1}$$



$\mathcal{G}_1 := (S_1, E_1)$

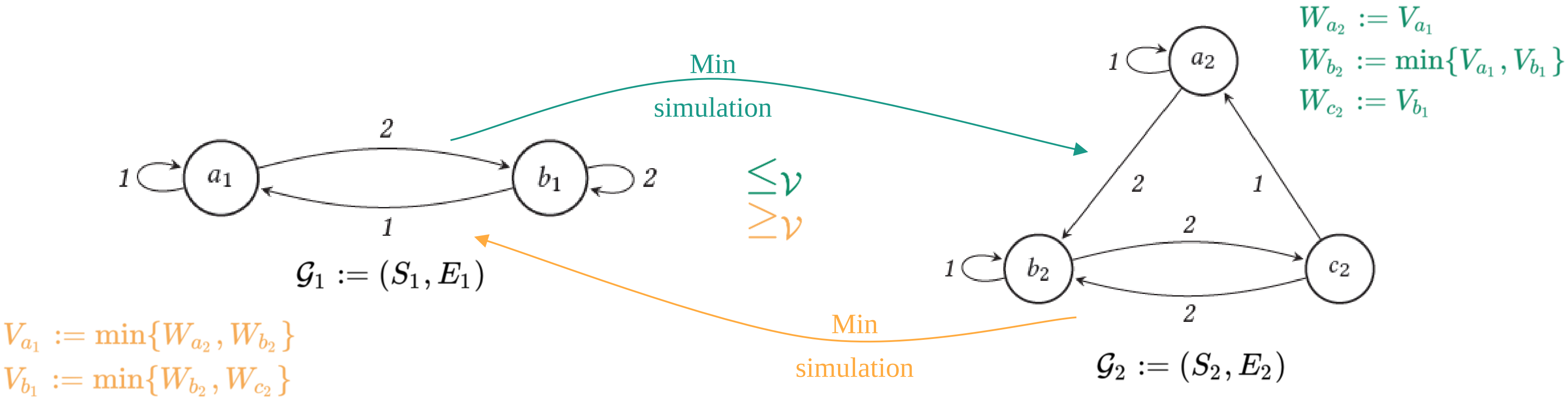
$\mathcal{G}_2 := (S_2, E_2)$

5. Numerical example *Approximation of the JSR*

Given a **linear switched system** with two modes

$$A_1 = \begin{bmatrix} 0.9 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0.6 & 0.9 \\ 0.6 & 0.3 \end{bmatrix}$$

We want to approximate the **joint spectral radius** (JSR) of $\mathcal{A} := \{A_1, A_2\}$.



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$\gamma_{\mathcal{G},\mathcal{V}}(\mathcal{A})$		**Template**	
		Copositive norms $V_s(x) := v_s^\top x$	***Quadratics*** $V_s(x) := x^\top P_s x$
Path-complete Graph	$\mathcal{G}_1 := (S_1, E_1)$	1.549	
	$\mathcal{G}_2 := (S_2, E_2)$	1.549	

for template $\mathcal{V}$ closed under min

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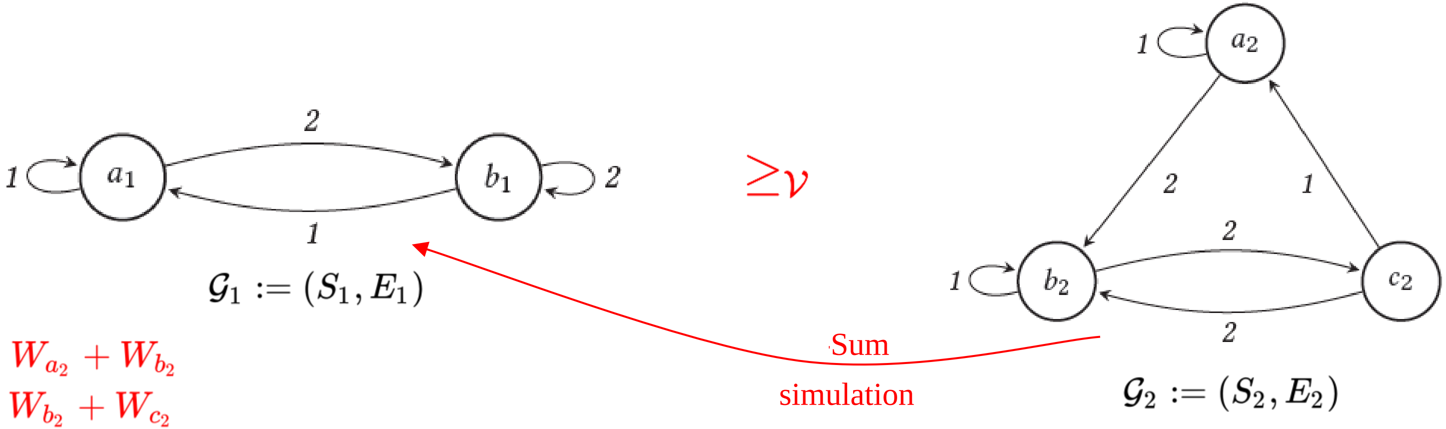

$$V_{a_1} := W_{a_2} + W_{b_2}$$
$$V_{b_1} := W_{b_2} + W_{c_2}$$

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		Copositive norms $V_s(x) := v_s^\top x$	***Quadratics*** $V_s(x) := x^\top P_s x$
Path-complete Graph	$\mathcal{G}_1 := (S_1, E_1)$	1.549	1.356
	$\mathcal{G}_2 := (S_2, E_2)$	1.549	1.364

for template $\mathcal{V}$ closed under sum

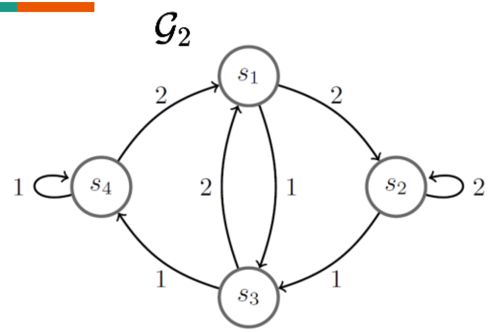
Presentation outline

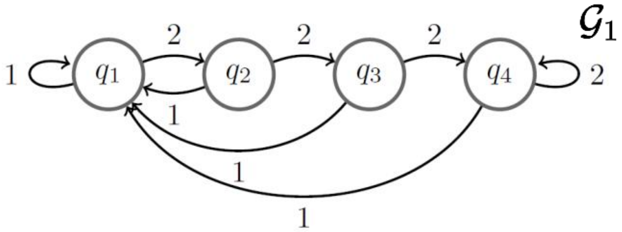
1. Introduction
2. The comparison problem
3. Template-dependent lifts
4. characterization theorems
5. Numerical example
6. **Two open problems**
7. Conclusion and future work

Statistical comparison

6. Statistical comparison

Some graphs seem to work better on average, but not always





Graph G1 'remembers' the last two symbols read

Graph G2 remembers the last symbol if it's a '1', the last three symbols if the last one is a '2'

What is the best strategy?
(i.e. which graph gives the best upper bound?)

6.Statistical comparison

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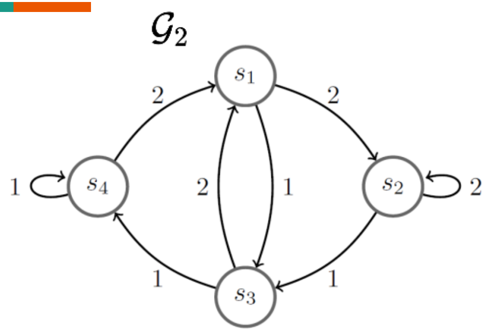

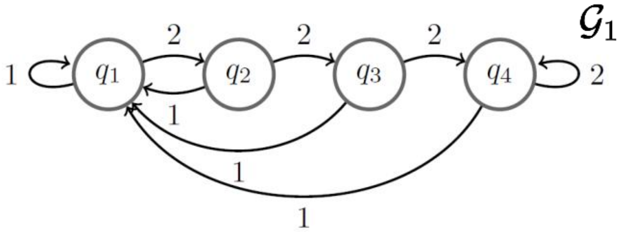

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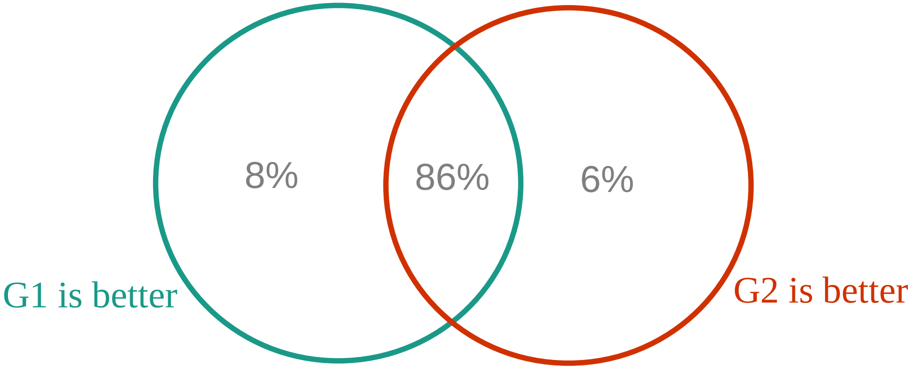

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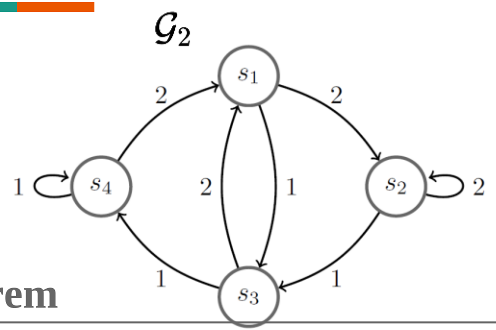

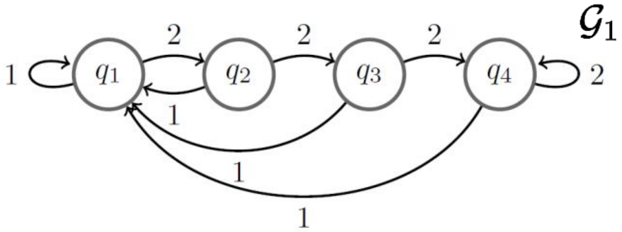

Theorem

Given two path-complete graphs $\mathcal{G}_1 := (S_1, E_1)$ and $\mathcal{G}_2 := (S_2, E_2)$. The following statements are equivalent :

A. $\mathcal{G}_1$ simulates $\mathcal{G}_2$.

B. $\mathcal{G}_1 \leq \mathcal{G}_2$.

6. Statistical comparison

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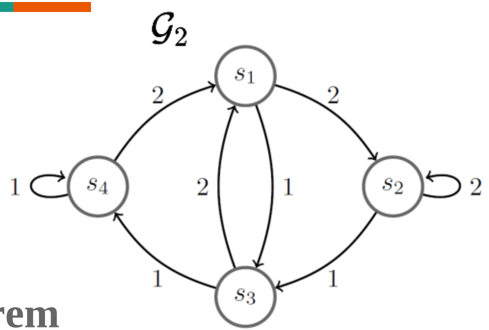
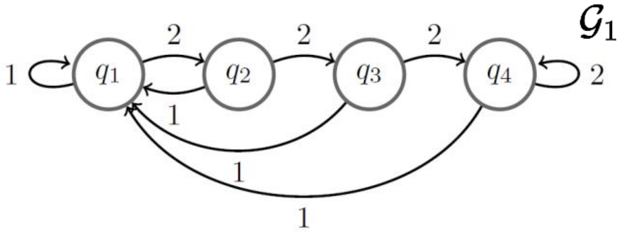




Theorem

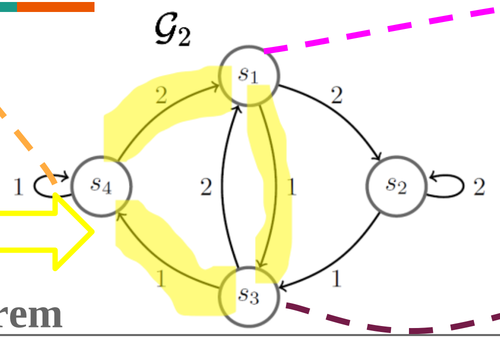
Given two path-complete graphs $\mathcal{G}_1 := (S_1, E_1)$ and $\mathcal{G}_2 := (S_2, E_2)$ **and a particular set of matrices**, if **a support set** of $\mathcal{G}_2$ is simulated in $\mathcal{G}_1$, then $\mathcal{G}_1 \leq \mathcal{G}_2$ **for this particular set of matrices**

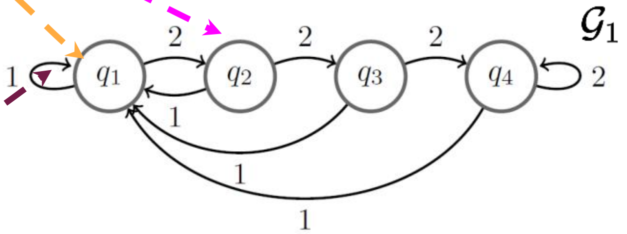
A *support set* of an optimization problem is a subset of constraints that defines the optimum (i.e. one can erase all the other constraints, the objective remains the same)

6. Statistical comparison

Some graphs seem to work better on average, but not always

Support set





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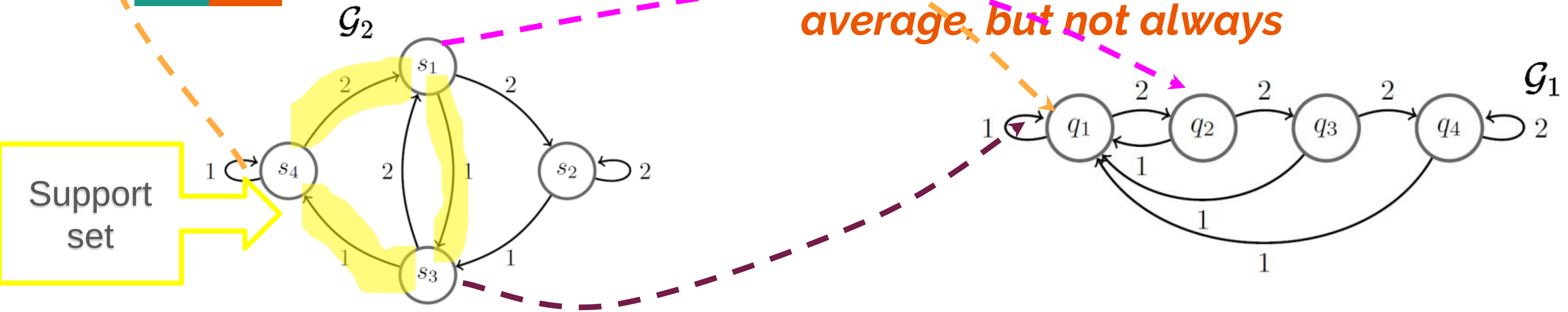


Theorem: $\mathcal{G}_2$ is better than $\mathcal{G}_1$ for every system for which the support set in $\mathcal{G}_2$ is simulated by $\mathcal{G}_1$

But

- How 'often' does that happen?
- How often does a particular set appear to be a support set?

➔ some support sets are more important than others!

➔ all this depends on the measure, on top of the template

Presentation outline

$$V(Ax) = (Ax)^T P (Ax)$$
$$= x^T A^T P A x$$

Quadratic PCLF and composition-closed templates

$$V_i(x) = x^T P_i x$$

$$(V_1 + V_2)(x) = x^T \underbrace{(P_1 + P_2)} x$$

$\Rightarrow$ if G_1^{sum} simulates G_2

$\rightarrow G_1 \leq_Q G_2$

Path-complete Lyapunov functions

Quadratic Lyapunov functions are closed under sum …

And thus **the sum-lift implies an ordering of quadratic PCLFs**

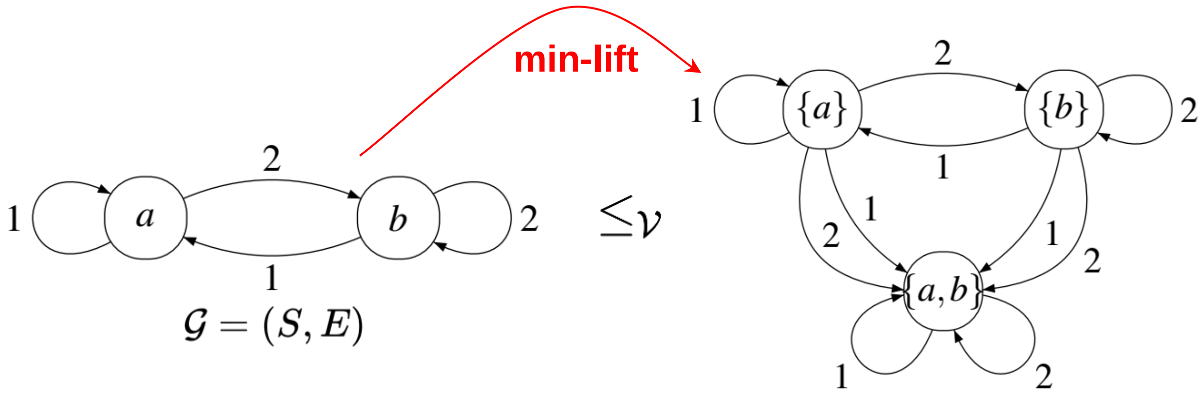
But also under composition by linear operators!

3. Template-dependent lifts

Example

Consider a template $\mathcal{V}$ closed under **min** :

$$W_{\{a\}} := V_a$$
$$W_{\{b\}} := V_b$$
$$W_{\{a,b\}} := \min\{V_a, V_b\} \in \mathcal{V}$$



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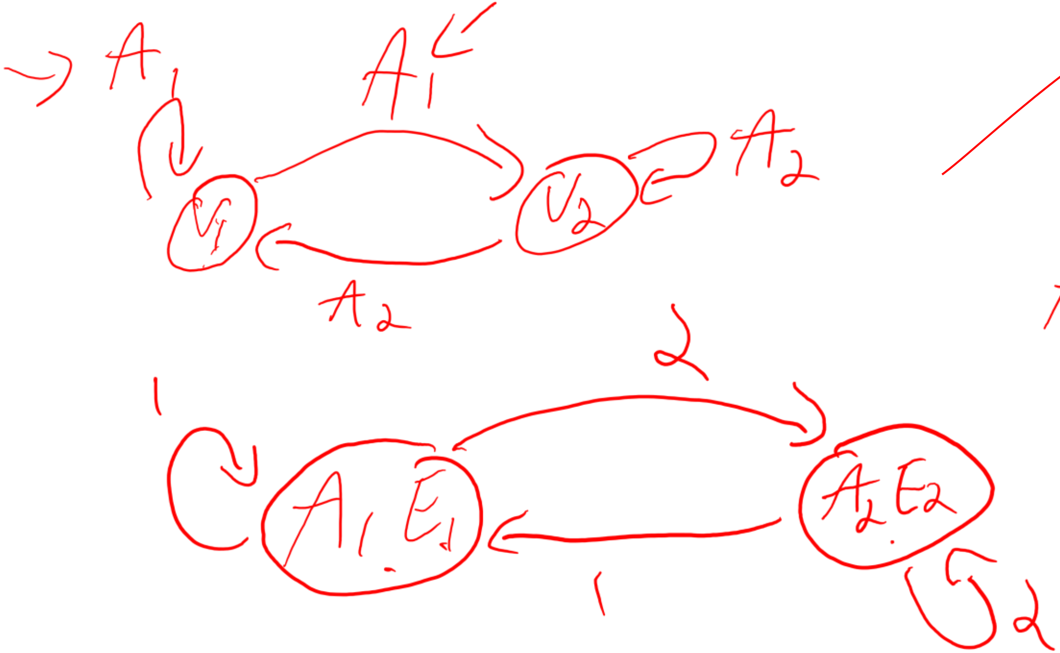

comp-lift

$A_1 \quad A_1E_1 \quad CA_1E_1$

$A_2 \quad A_1E_1 \; C \; A_2E_2$

became $A_1E_1 \; C \; E_2$

3.Template-dependent lifts *Composition-lift*

We say that $L : Graphs_M \rightarrow Graphs_M$ is a **valid lift with respect to a template** $\mathcal{V}$ if

(1) $\mathcal{G}$ path-complete implies $L(\mathcal{G})$ is path-complete,

(2) $\mathcal{G} \leq_{\mathcal{V}} L(\mathcal{G})$, for all path-complete graph $\mathcal{G}$.

Theorem: If the composition lift of $\mathcal{G}_1$ simulates $\mathcal{G}_2$, then $\mathcal{G}_1 \leq_{\mathcal{Q}} \mathcal{G}_2$

But what about the converse?

- Is the composition lift infinite?
- Is it sufficient to describe the composition order?
- Is it sufficient to describe the quadratic order (together with the sum lift)?

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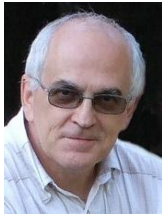
Conclusion: a perspective on switching systems

[Furstenberg Kesten, 1960]

[Gurvits, 1995]



[Kozyakin, 1990]

[Daafouz Bernussou, 2002]

[Rantzer Johansson 1998]

[Rota, Strang, 1960]

[Blondel Tsitsiklis, 98+]

[Lee Dullerud 2006]

[Parrilo Jadbabaie 20…

60s 70s

Mathematical properties

90s

TCS inspired
Negative
Complexity results

2000s

Lyapunov/LMI Techniques (S-procedure)

now

CPS applications ⇔ Smart algorithms

7.Conclusion *And future work*

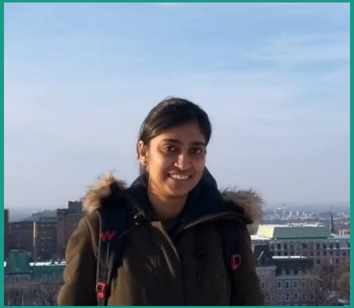
- We have provided **combinatorial tools** to analyze the **performance of algebraic optimization programs** for control (here: stability analysis of switched systems)
- These results **exploit and mix algebraic properties** of the template of Lyapunov functions and **combinatorial properties of the automata**
- Possibility to massively **generalize these concepts** to more general control problems, and more general systems (in progress)
- Many **open questions** left!

Long term goal:

Develop tailored optimization programs leveraging the properties of the system, for general control purpose

Thanks for your attention!

And thanks to my co-authors



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