

Semilinear and rational sets with data

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The family R of **rational sets** of a monoid $(M, +)$ is the smallest family that contains all singletons and is closed under

- union

$$A \in R, B \in R \implies A \cup B \in R$$

- operation of the monoid

$$A \in R, B \in R \implies A + B = \{a + b : a \in A, b \in B\} \in R$$

- Kleene star

$$A \in R \implies A^* = \{a_1 + \dots + a_n : n \in \mathbb{N}, a_1, \dots, a_n \in A\} \in R$$

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Example: regular languages

$$(\mathbf{N}^d, +)$$

Rational sets

$(\mathbf{N}^d, +)$

Rational sets



Semilinear sets

Rational sets of star height 1



Finite unions of linear sets

$b + P^*$

$(\mathbf{N}^d, +)$

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Commutative images
of regular languages

$a(ab)^*a \mapsto (2,0) + \{(1,1)\}^*$

Parikh Theorem

$(\mathbb{N}^d, +)$

Rational sets



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Finite unions of linear sets

$b + P^*$



Sets definable
in Presburger arithmetic

First order theory of $(\mathbb{N}, +)$

Commutative images
of regular languages

$a(ab)^*a \mapsto (2,0) + \{(1,1)\}^*$

Parikh Theorem

Let $\mathbb{A} = \{ \text{orange circle} , \text{red circle} , \text{blue circle} , \text{green circle} , \dots \}$ be an infinite set of data.

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A **data vector** is a finitely supported vector from $\mathbb{N}^{d \times \mathbb{A}}$

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Permutations of \mathbb{A} act on data vectors

$$(2 \text{orange circle} + \text{green circle}, 3 \text{blue circle} + 5 \text{orange circle}, \text{blue circle}) \mapsto (2 \text{pink circle} + \text{red circle}, 3 \text{blue circle} + 5 \text{pink circle}, \text{blue circle})$$

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A set is **orbit-finite** if it can be split into finitely many orbits.

The family R of **rational sets** of $(\mathbb{N}_{fin}^{d \times \mathbb{A}}, +)$ is the smallest family that contains all singletons and is closed under

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- orbit-finite unions

$$\forall_{i \in I} A_i \in R \implies \bigcup_{i \in I} A_i \in R, \quad \text{for any orbit-finite set } I$$

- addition

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Example

$$S = \bigcup_{a \neq b \in \mathbb{A}} ((a, b) + \{(a, 2a)\}^*)$$

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Example

$$S = \bigcup_{a \neq b \in \mathbb{A}} ((a, b) + \{(a, 2a)\}^*)$$

$$(2 \text{ orange}, \text{ green} + 2 \text{ orange}) \in S$$

$$(2 \text{ blue}, \text{ red} + 2 \text{ blue}) \in S$$

$$(3 \text{ blue}, \text{ red} + 4 \text{ blue}) \in S$$



Rational sets



A Venn diagram consisting of two concentric semi-circles. The outer semi-circle is light purple and has a dashed border. The inner semi-circle is light orange and has a solid border. The inner semi-circle is entirely contained within the outer one. The text 'Rational sets' is centered in the purple region, and 'Semilinear sets' is centered in the orange region. Below 'Semilinear sets' is the text 'Rational sets of star height 1'.

Rational sets

Semilinear sets

Rational sets of star height 1



A Venn diagram consisting of two concentric circles. The outer circle is light purple and contains the text 'Rational sets'. The inner circle is light orange and contains the text 'Semilinear sets' and 'Rational sets of star height 1'. The inner circle is entirely contained within the outer circle. A dashed line separates the two circles. The symbol \cup is placed between the two circles.

Rational sets

\cup

[Hofman, Juzepczuk, Lasota, Pattathurajan, '21]

Semilinear sets

Rational sets of star height 1



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Rational sets

\cup

[Hofman, Juzepczuk, Lasota, Pattathurajan, '21]

Even for $d = 1$

Semilinear sets

Rational sets of star height 1

Commutative images
of languages of NRA

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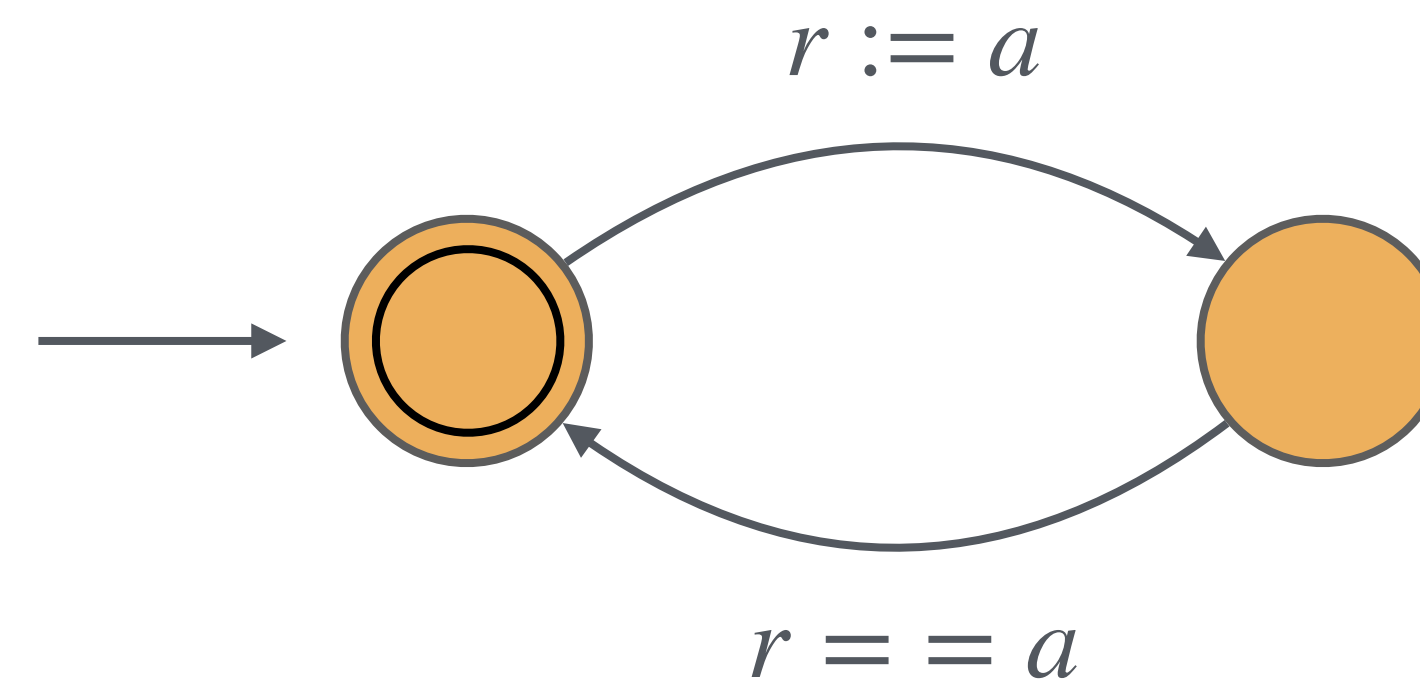
\cup

Semilinear sets

Rational sets of star height 1

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Commutative images
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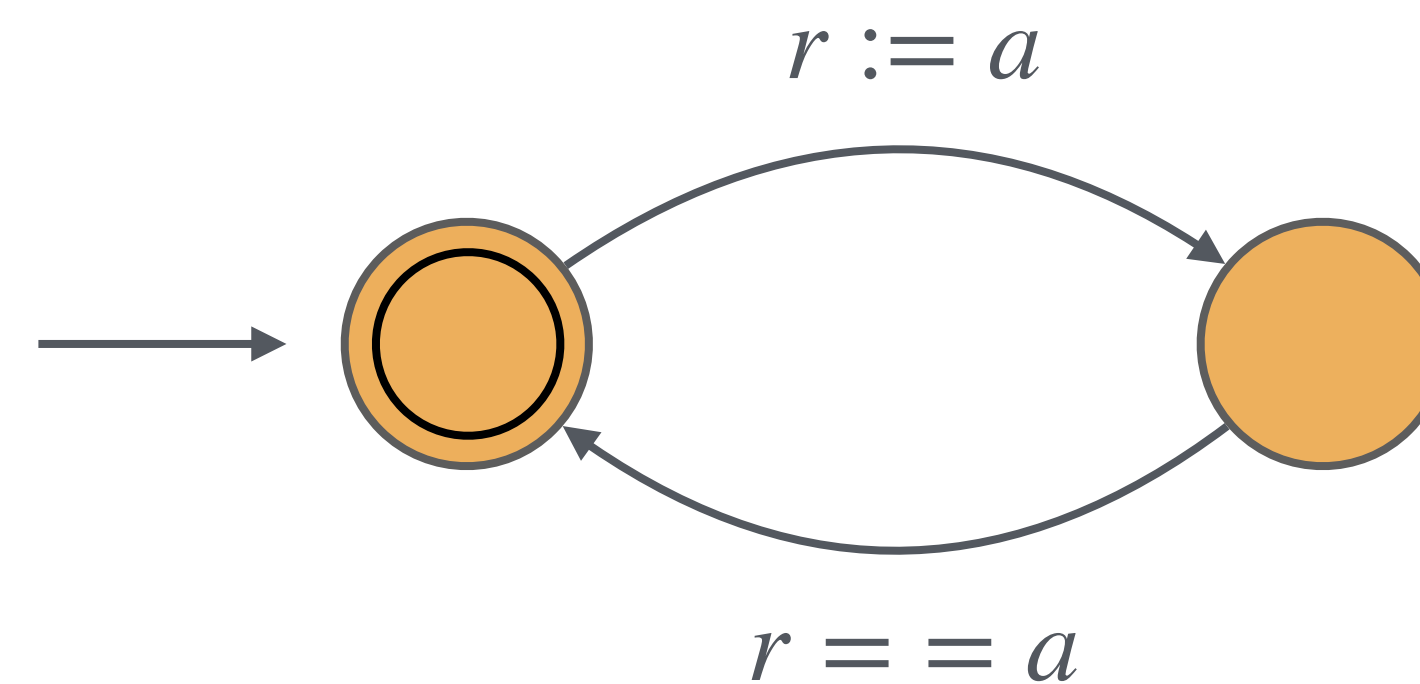
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Semilinear sets

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[Hofman, Juzepczuk, Lasota, Pattathurajan, '21]

Even for $d = 1$



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Commutative images
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Rational sets

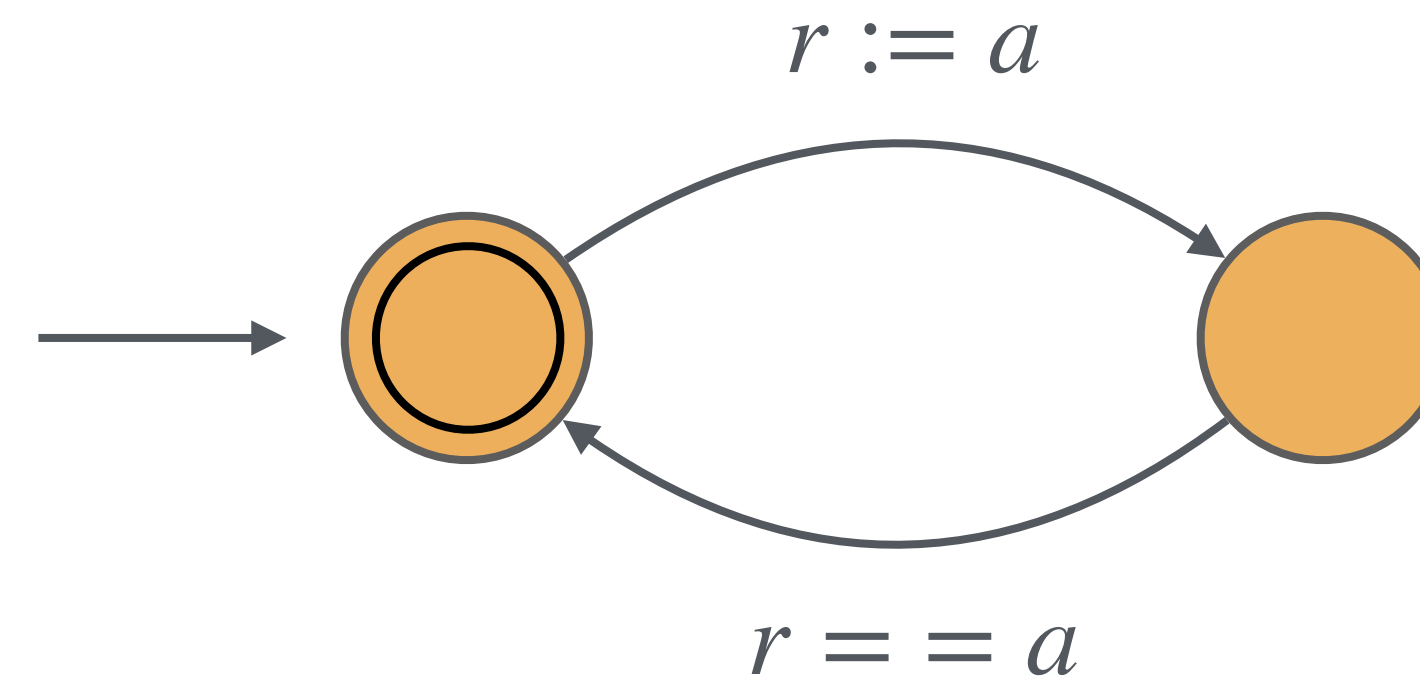
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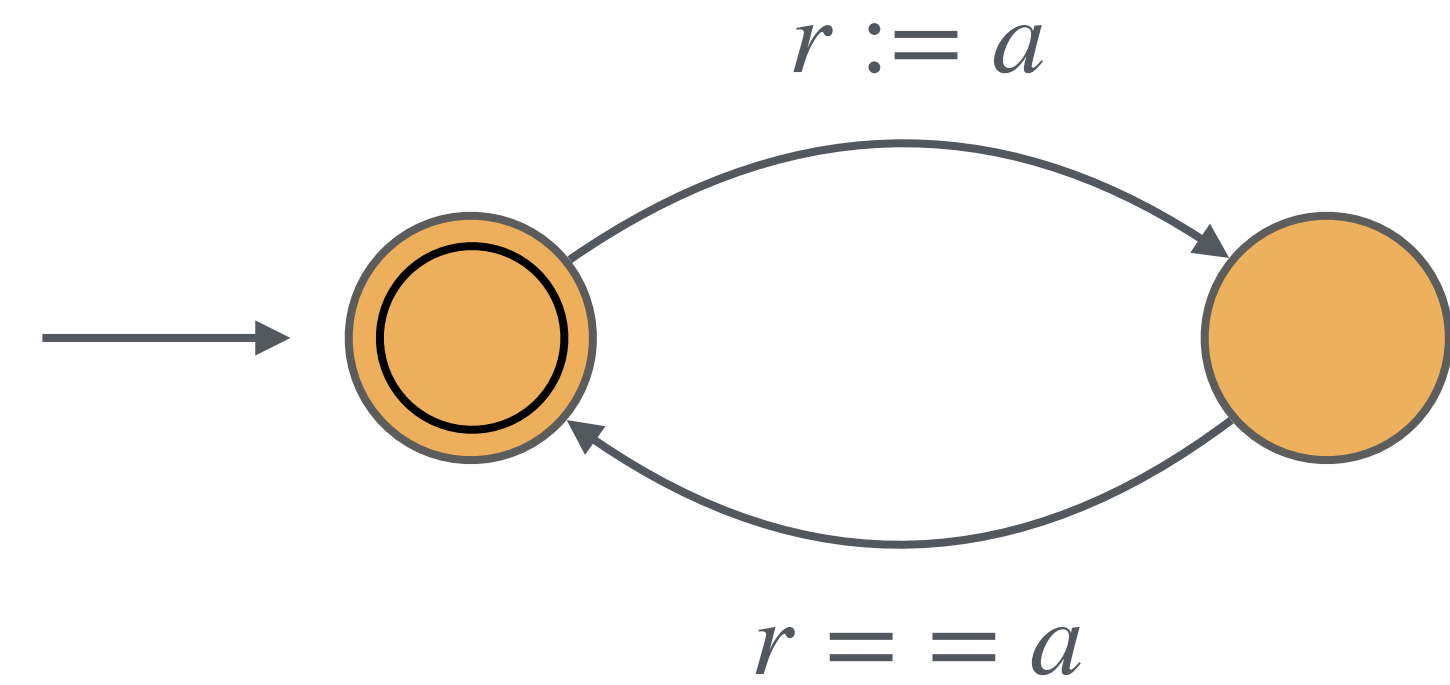
Semilinear sets

Rational sets of star height 1

Open problem:
is it equality?

[Hofman, Juzepczuk, Lasota, Pattathurajan, '21]

Even for $d = 1$



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Open problems

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- Are commutative images of languages of RNA rational?

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- Is there a number d such that all rational sets have star height d ?

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- Decision problems: membership, equality, emptiness of intersection

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Thank you!