Semilinear and rational sets with data

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SAMSA, Warsaw, 3rd June 2025

The family R of rational sets of a monoid $(M, +)$ is the smallest family that contains all singletons and is closed under

- union

$$A \in R,\ B \in R \implies A \cup B \in R$$

- operation of the monoid

$$A \in R,\ B \in R \implies A + B = \{a + b : a \in A,\ b \in B\} \in R$$

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$$A \in R \implies A^* = \{a_1 + \ldots + a_n : n \in \mathbb{N},\ a_1, \ldots, a_n \in A\} \in R$$

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Example: regular languages

$(\mathbf{N}^d, +)$

Rational sets

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Rational sets

$=$

Semilinear sets

Rational sets of star height 1

$||$

Finite unions of linear sets

$b + P^*$

$(\mathbf{N}^d, +)$

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Commutative images of regular languages

$a(ab)^*a \quad \mapsto \quad (2,0) + \{(1,1)\}^*$

Parikh Theorem

$(\mathbf{N}^d, +)$

Rational sets

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Semilinear sets

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Sets definable in Presburger arithmetic

First order theory of $(\mathbb{N}, +)$

=

Commutative images of regular languages

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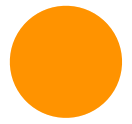
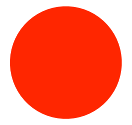
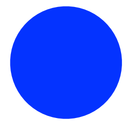
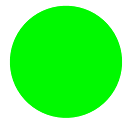
Parikh Theorem

Let $\mathbb{A} = \{$  $,$  $,$  $,$  $, \ldots \}$ be an infinite set of data.

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A data vector is a finitely supported vector from $\mathbb{N}^{d \times \mathbb{A}}$

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Example: (2🟠 + 🟢, 3🔵 + 5🟠, 🔵)

Let $\mathbb{A} = \{ \textcolor{orange}{\bullet}, \textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{green}{\bullet}, \ldots \}$ be an infinite set of data.

A data vector is a finitely supported vector from $\mathbb{N}^{d \times \mathbb{A}}$

Example: $(2\textcolor{orange}{\bullet} + \textcolor{green}{\bullet},\ 3\textcolor{blue}{\bullet} + 5\textcolor{orange}{\bullet},\ \textcolor{blue}{\bullet})$

Permutations of $\mathbb{A}$ act on data vectors

$$(2\textcolor{orange}{\bullet} + \textcolor{green}{\bullet},\ 3\textcolor{blue}{\bullet} + 5\textcolor{orange}{\bullet},\ \textcolor{blue}{\bullet}) \quad \mapsto \quad (2\textcolor{magenta}{\bullet} + \textcolor{red}{\bullet},\ 3\textcolor{blue}{\bullet} + 5\textcolor{magenta}{\bullet},\ \textcolor{blue}{\bullet})$$

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A set is orbit-finite if it can be split into finitely many orbits.

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- orbit-finite unions

$$\forall_{i\in I}\ A_i \in R \implies \bigcup_{i\in I} A_i \in R, \quad \text{for any orbit-finite set } I$$

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Example

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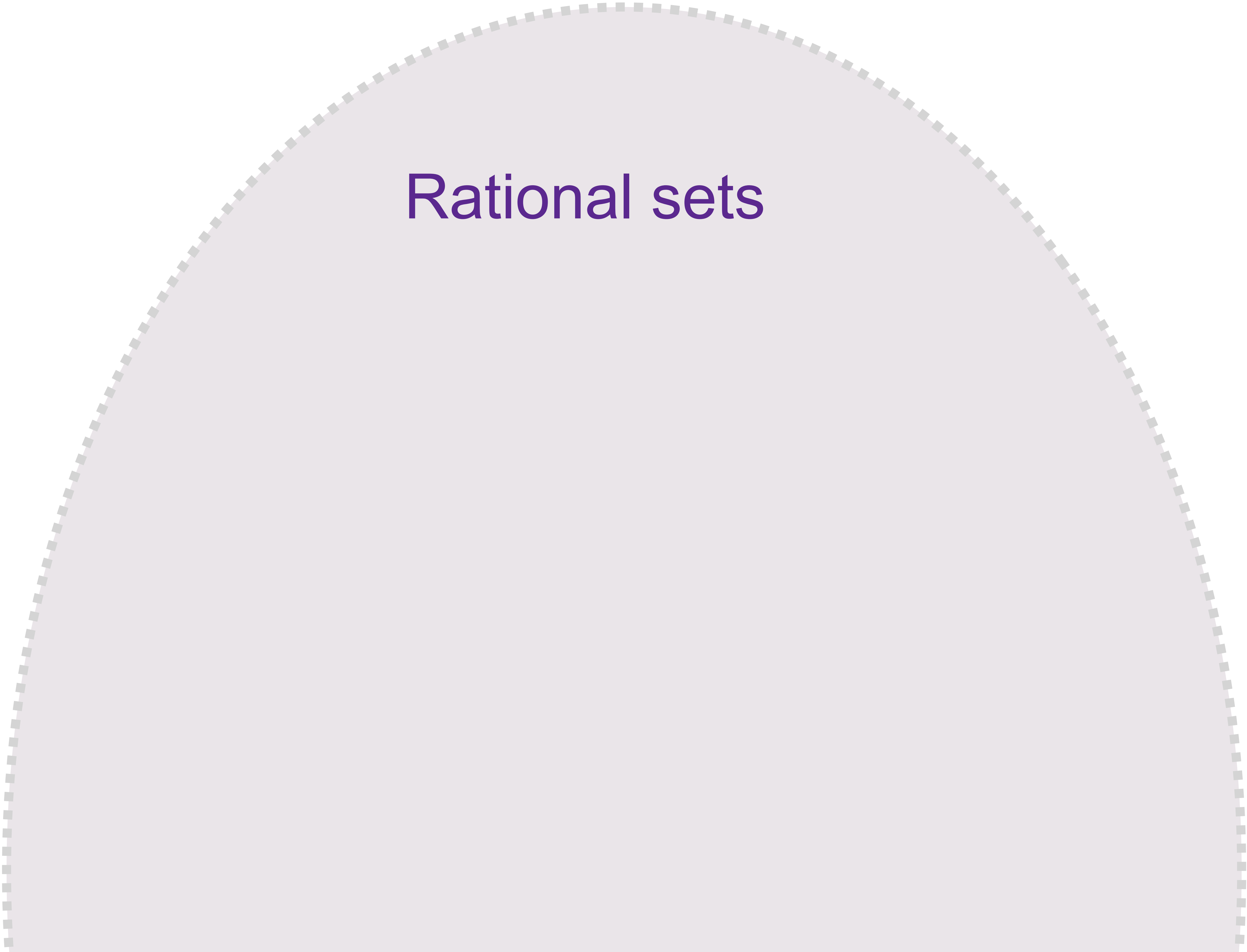
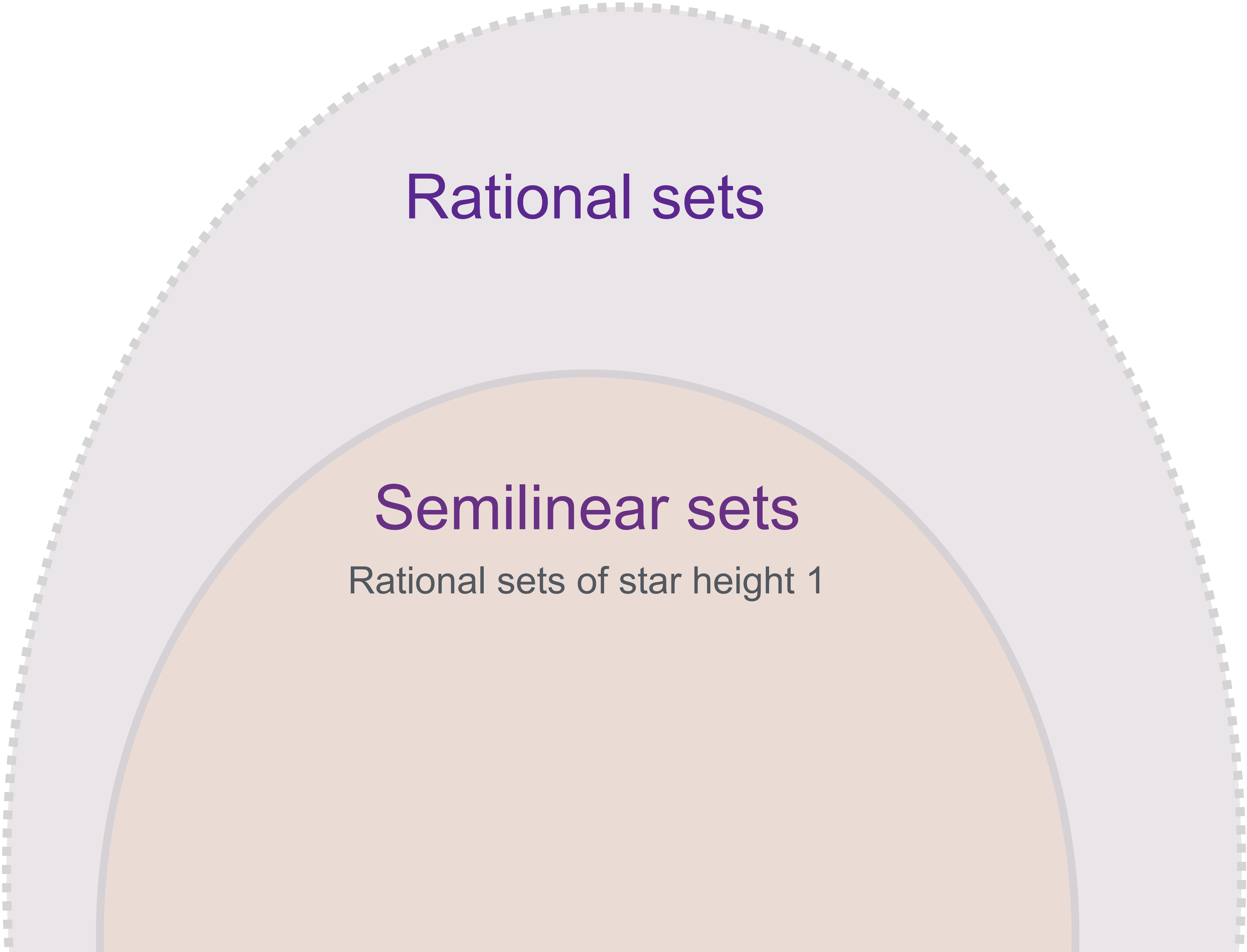
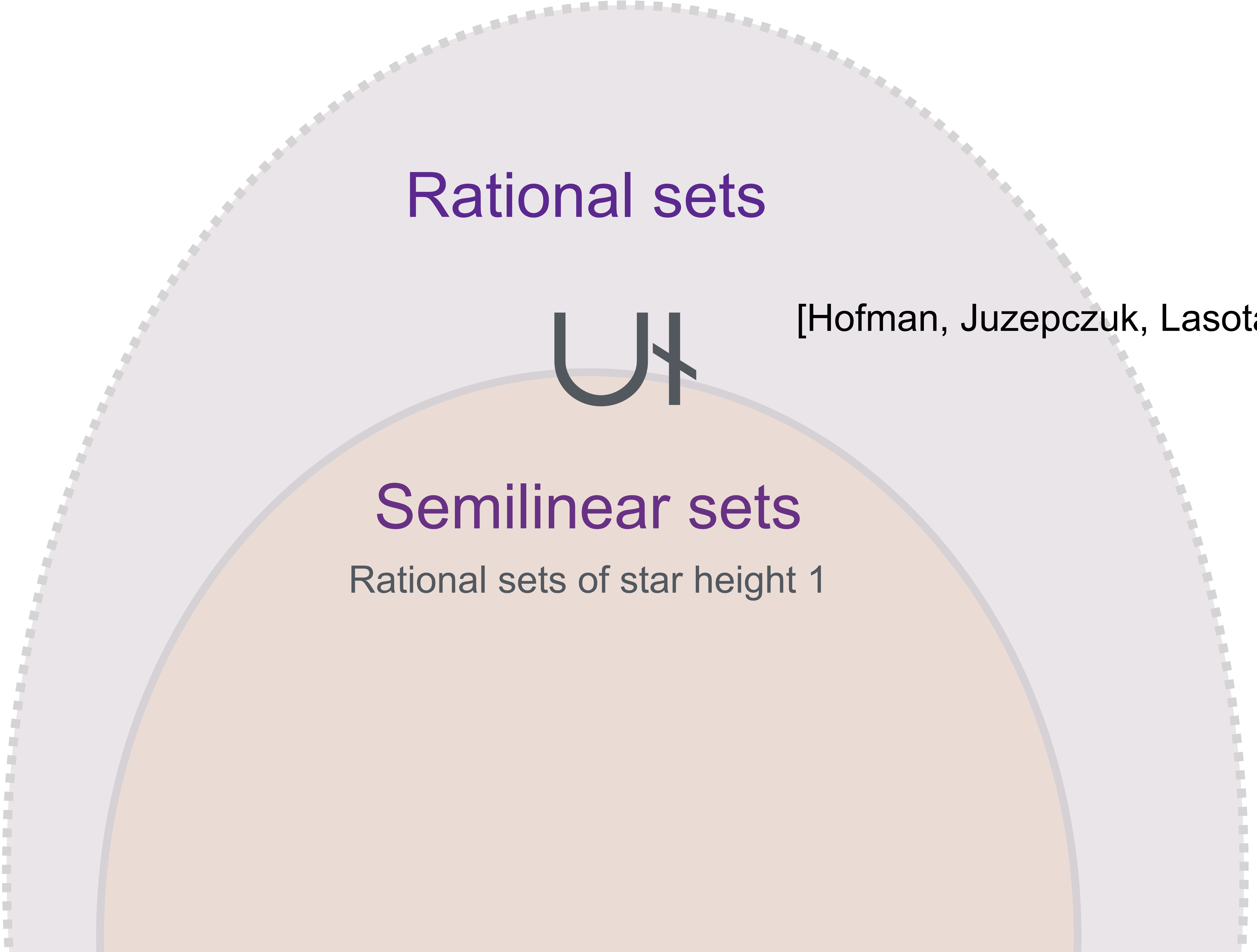
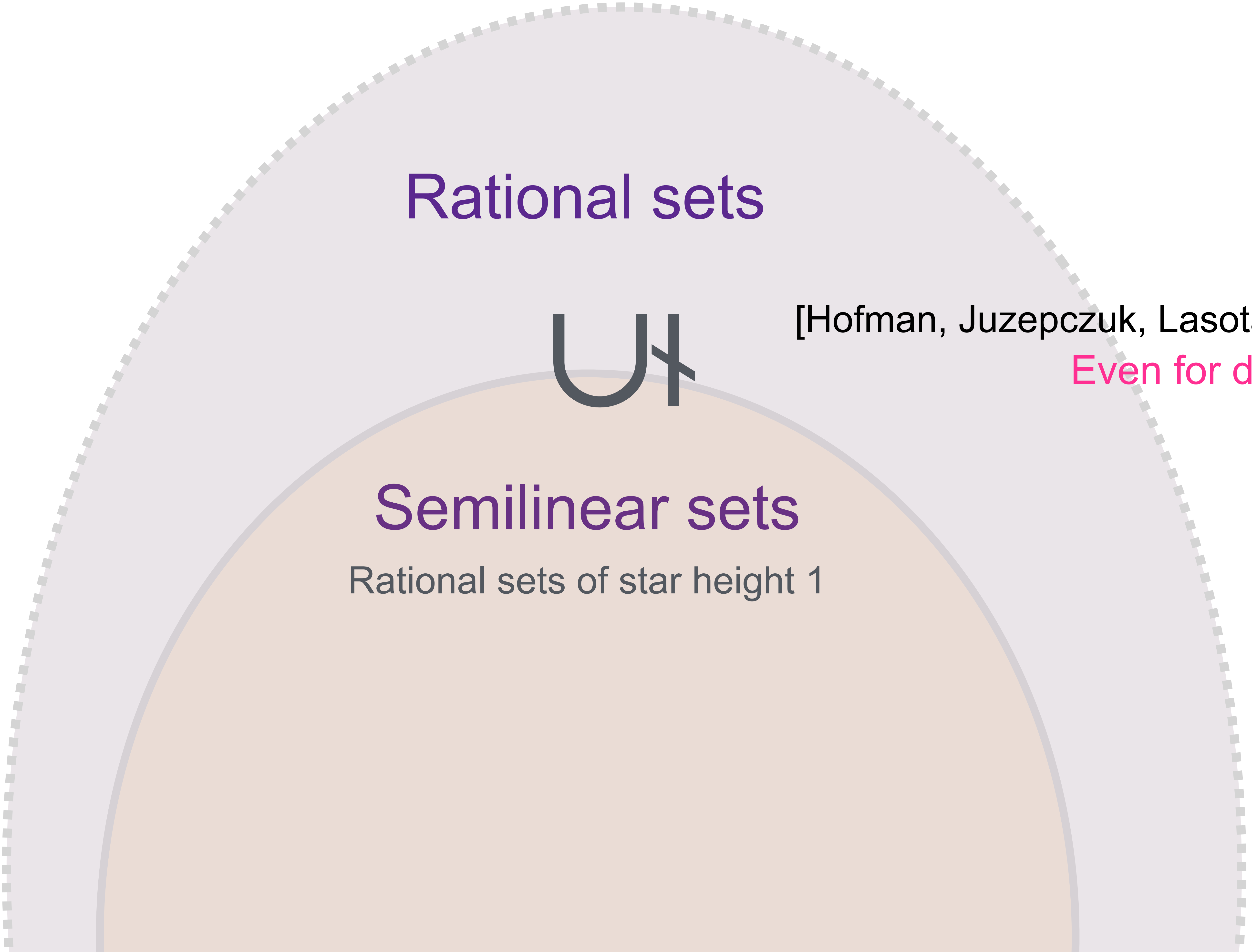
$$(3{\color{blue}\bullet},\ {\color{red}\bullet} + 4{\color{blue}\bullet}) \in S$$


Rational sets


Rational sets
Semilinear sets
Rational sets of star height 1



[Hofman, Juzepczuk, Lasota, Pattathurajan, ’21]



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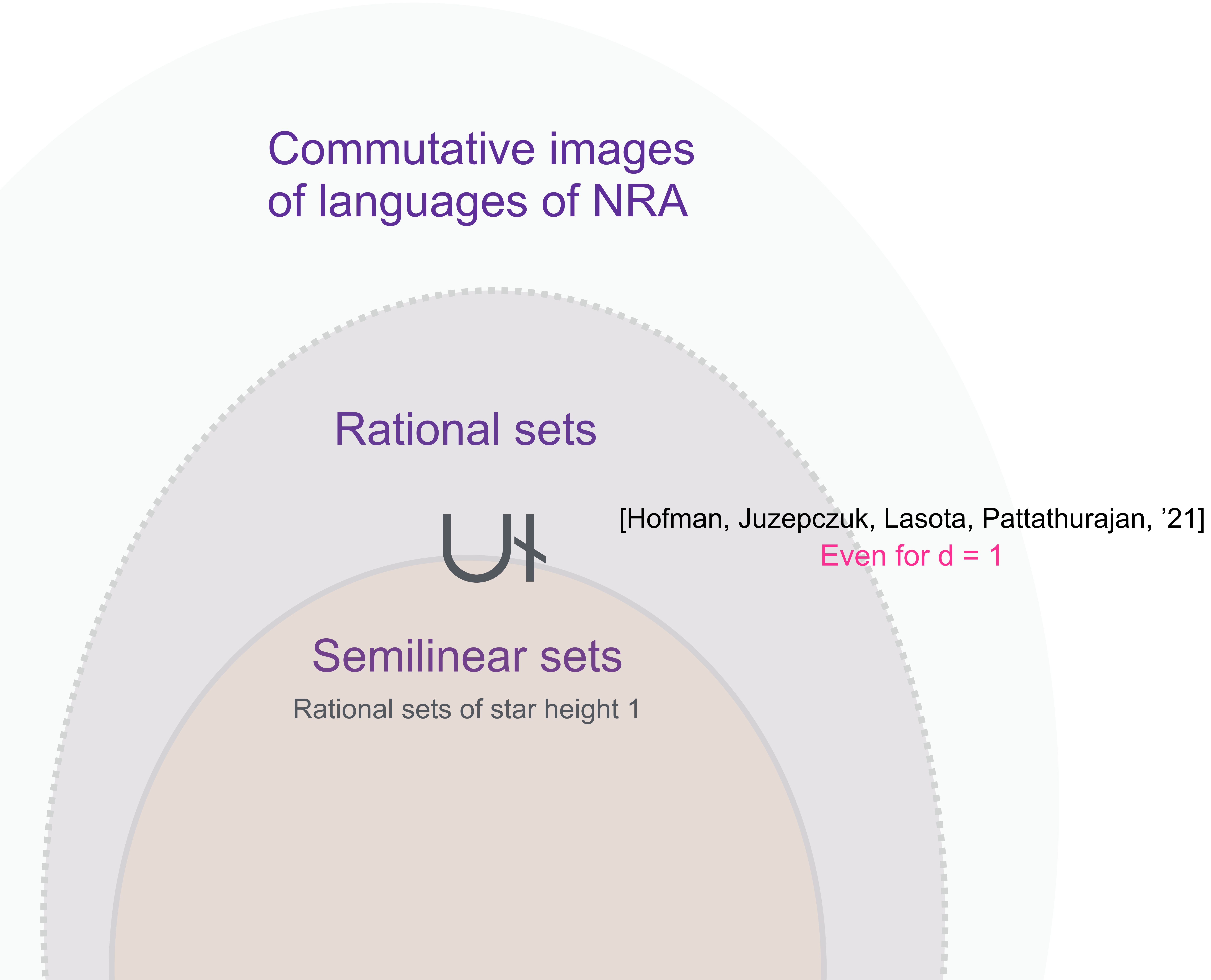
Even for d = 1


Commutative images
of languages of NRA
Rational sets
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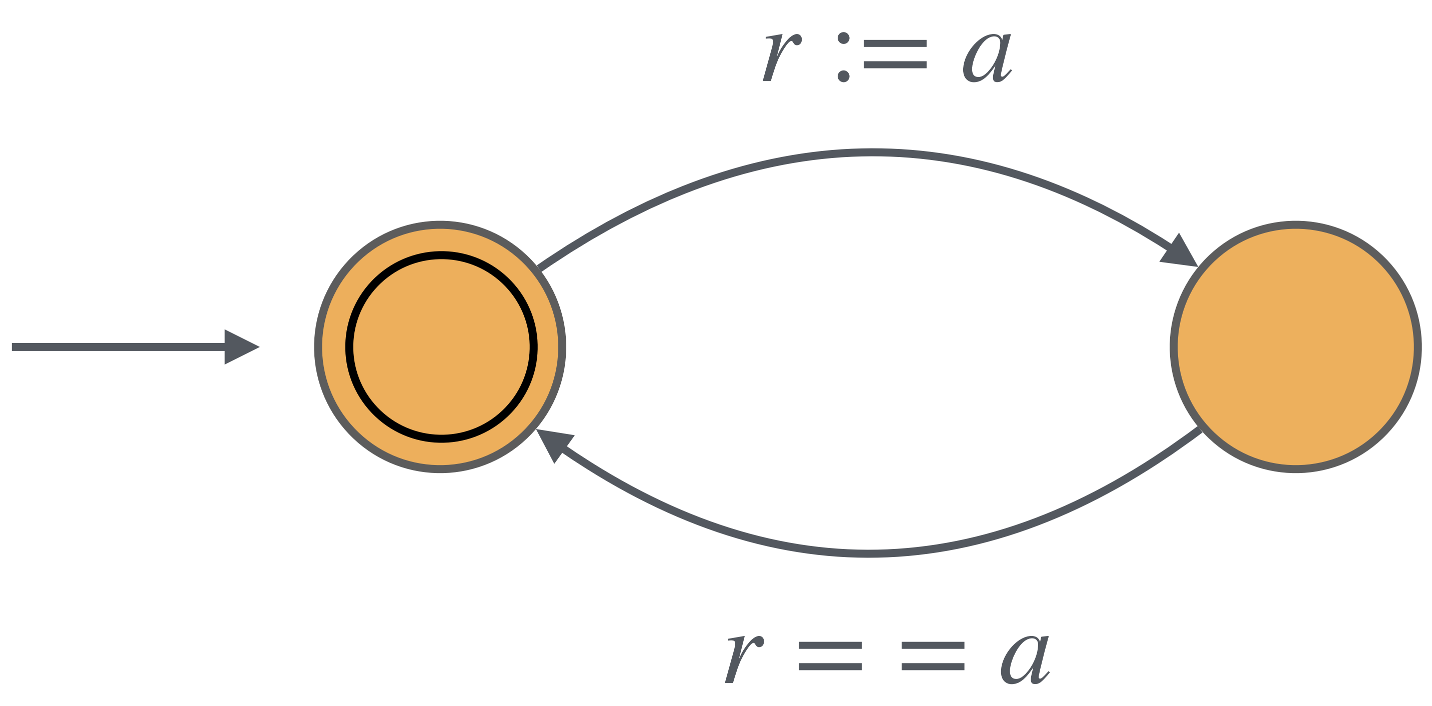

Commutative images of languages of NRA

Rational sets

$\supsetneq$

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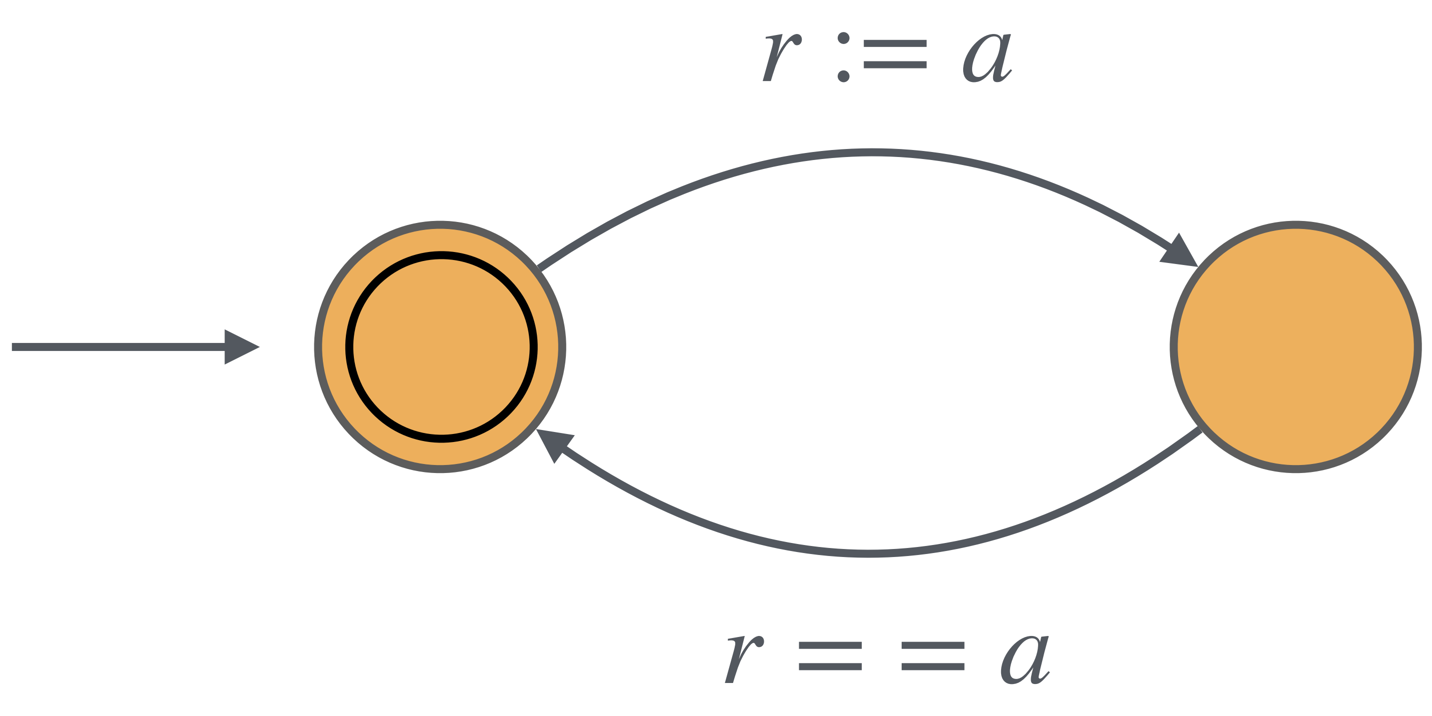

$\mapsto$

$$\left(\bigcup_{a\in\mathbb{A}} \{2a\}\right)^*$$

Commutative images of languages of NRA

$\cup\!|$

Rational sets

$\not\subseteq$ [Hofman, Juzepczuk, Lasota, Pattathurajan, '21]

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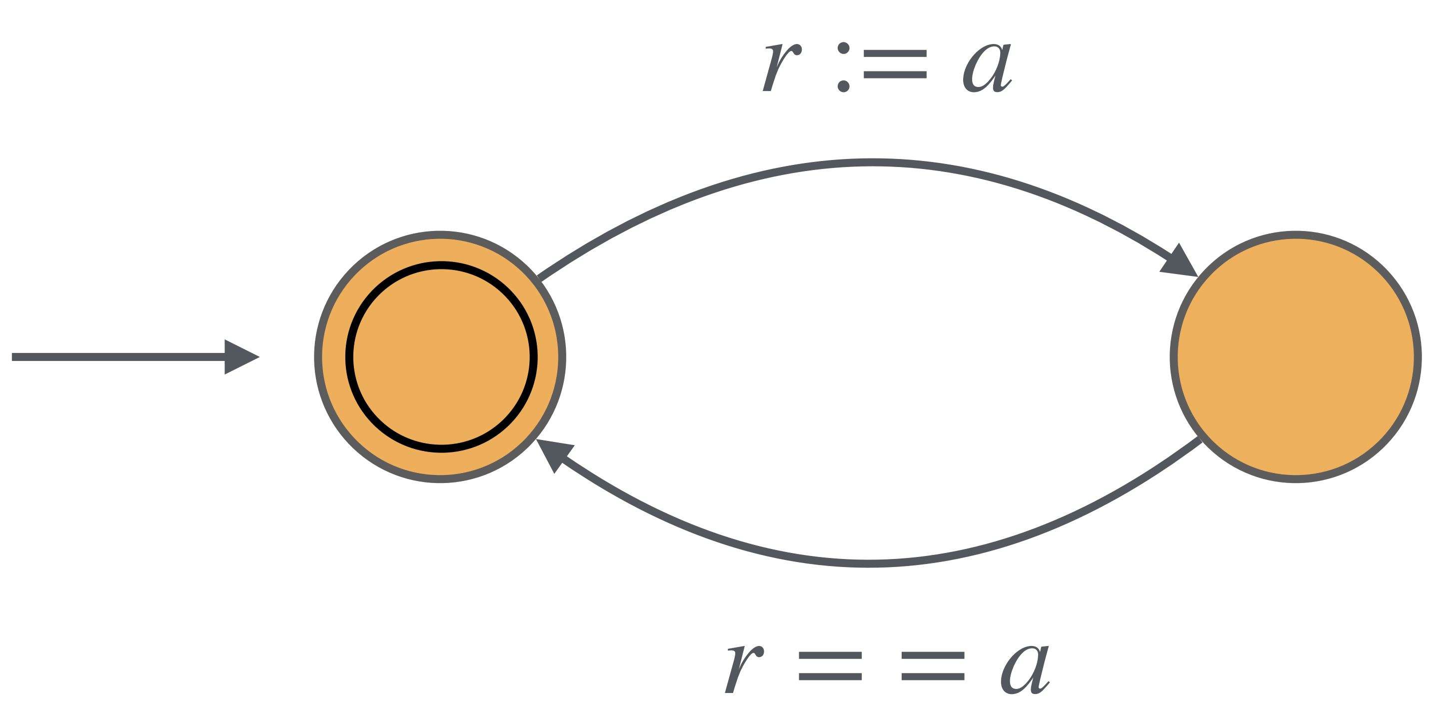

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Open problem: is it equality?

Rational sets

⊊

[Hofman, Juzepczuk, Lasota, Pattathurajan, '21]

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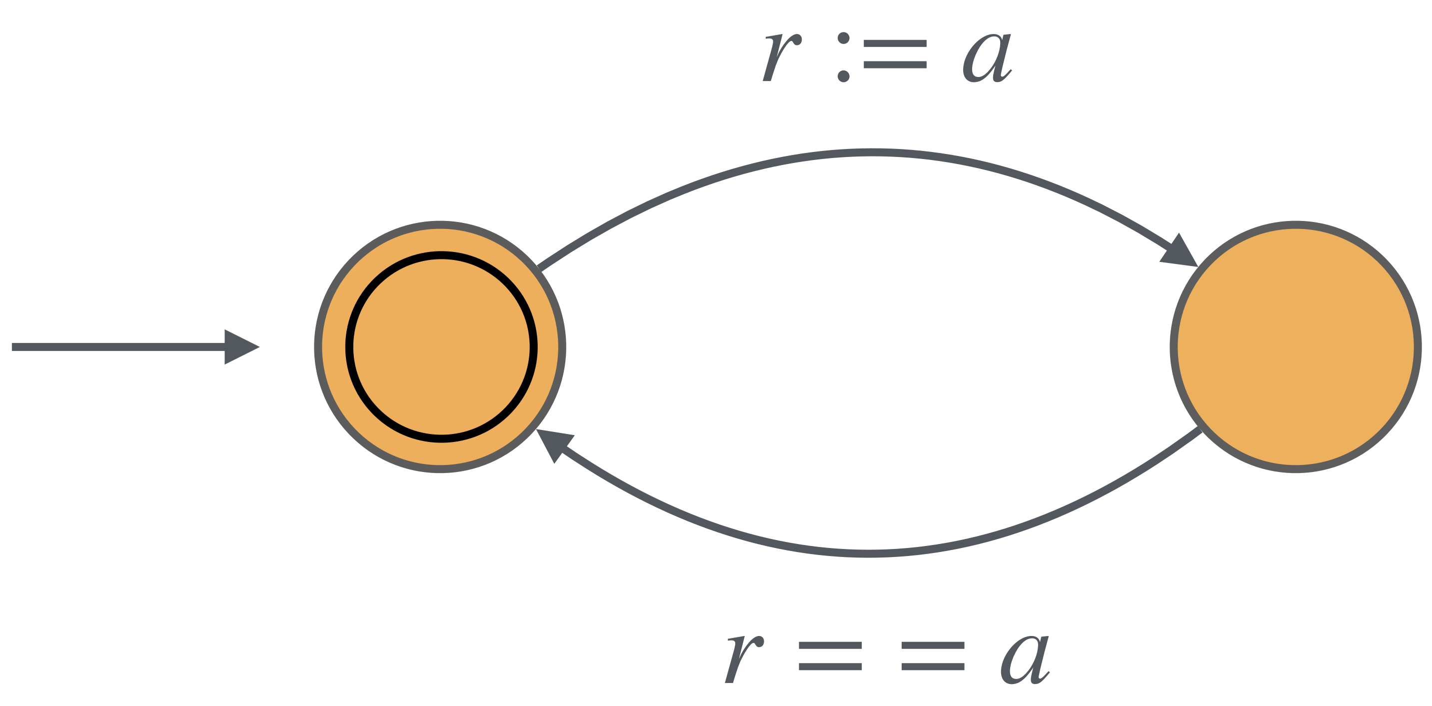

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Thank you!