03/06/2025

SAMSA Workshop

On the complexity of computing the linear hull of weighted automata over $\mathbb{Q}$

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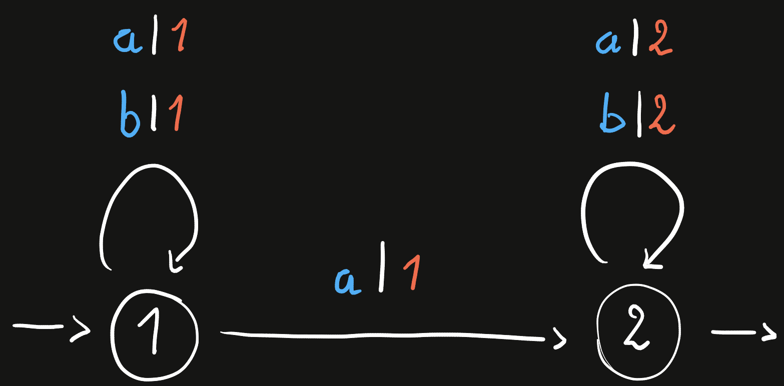
Socialement engagée

LIS
LABORATOIRE D'INFORMATIQUE & DES SYSTÈMES

Weighted Automata
(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

Weighted Automata

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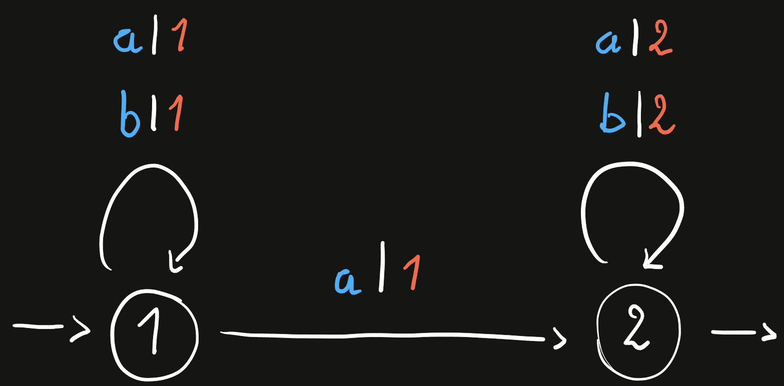

realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

aab :

$$w\left(1 \xrightarrow{a|1} 1 \xrightarrow{a|1} 2 \xrightarrow{b|2} 2 \right) = 1 \times 1 \times 2 = 2$$

Weighted Automata

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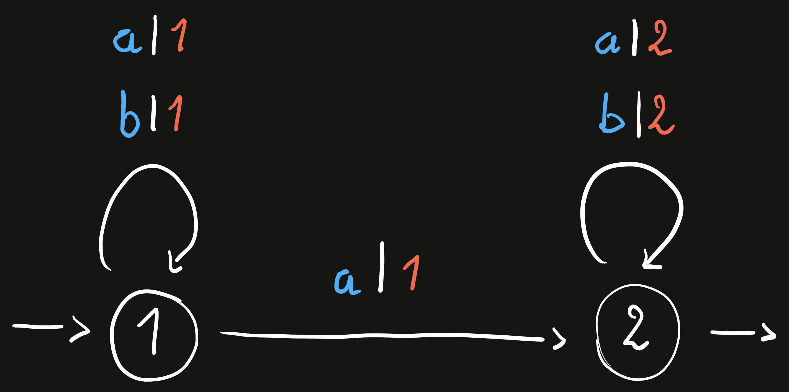

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$$w\left(① \xrightarrow{a|1} ② \xrightarrow{a|2} ② \xrightarrow{b|2} ② \right) = 1 \times 2 \times 2 = 4$$

Weighted Automata

(WA)

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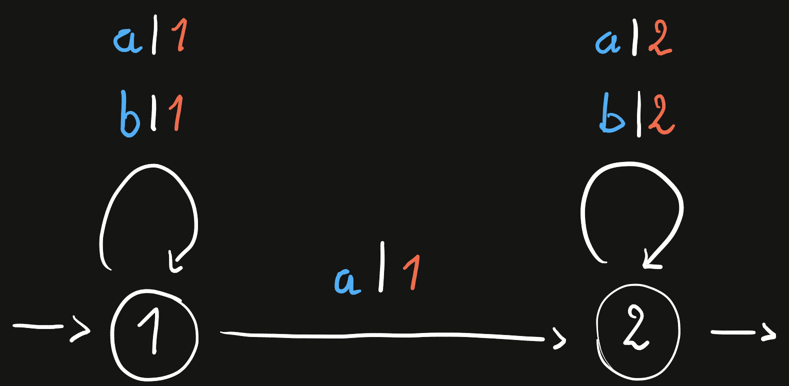

realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$$x_2 \longmapsto x_{10}$$

aab :

$$w\left(1 \xrightarrow{a|1} 1 \xrightarrow{a|1} 2 \xrightarrow{b|2} 2\right) = 1 \times 1 \times 2 = 2$$

$$+$$

$$w\left(1 \xrightarrow{a|1} 2 \xrightarrow{a|2} 2 \xrightarrow{b|2} 2\right) = 1 \times 2 \times 2 = 4$$

$$aab \mapsto 6$$

Weighted Automata

(WA)

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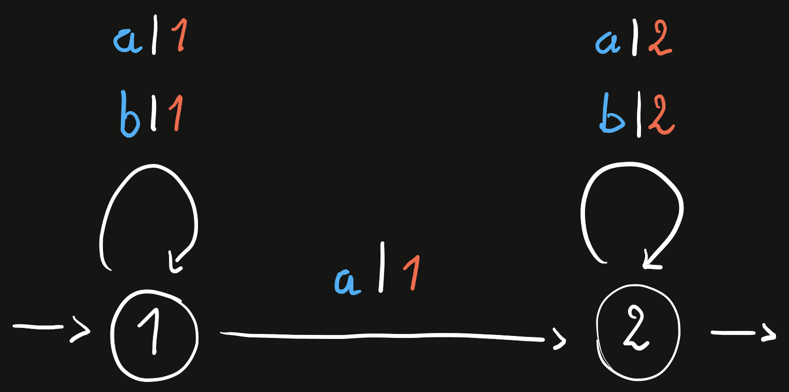

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$aab \mapsto 6$

Linear representation

(u, μ, v)

initial vector

$u = (\quad)$

terminal vector

$v = (\quad)$

transition matrices

$\mu(a) = (\quad)$ $\mu(b) = (\quad)$

Weighted Automata
(WA)

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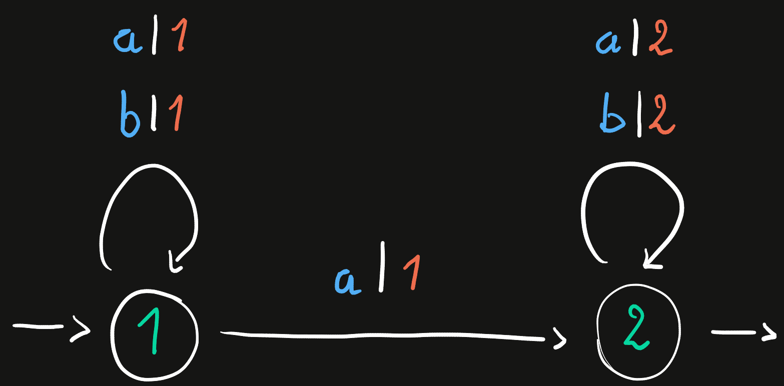

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$aab \mapsto 6$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \quad & \quad \end{pmatrix} \end{array}$$

terminal vector

$$v = \begin{array}{cc} \begin{pmatrix} \quad \\ \quad \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \end{array}$$

Weighted Automata
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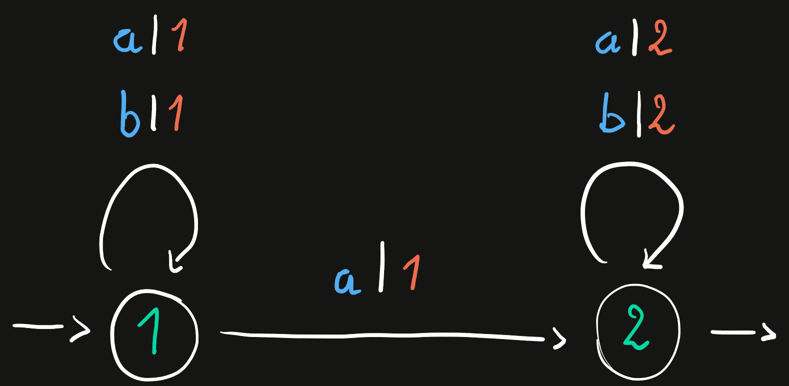

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$aab \mapsto 6$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$$

terminal vector

$$v = \begin{pmatrix} \\ \\ \end{pmatrix} \begin{array}{c} 1 \\ 2 \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \end{array}$$

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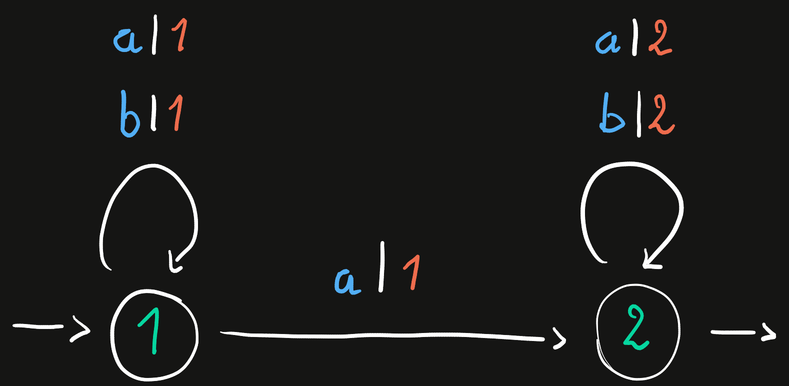

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$aab \mapsto 6$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{cc} 1 & 2 \\ \hline 1 & 0 \end{array}$$

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

terminal vector

$$v = \begin{array}{c|c} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} & \\ & \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} & \\ & \end{pmatrix} \end{array}$$

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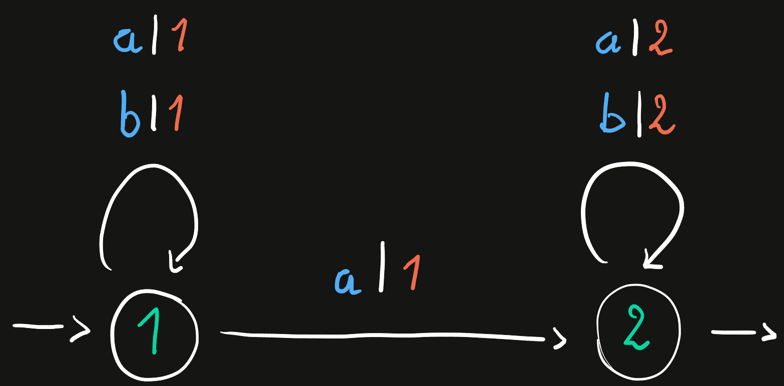

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Linear representation
(u, μ, v)

initial vector

$$u = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$$

terminal vector

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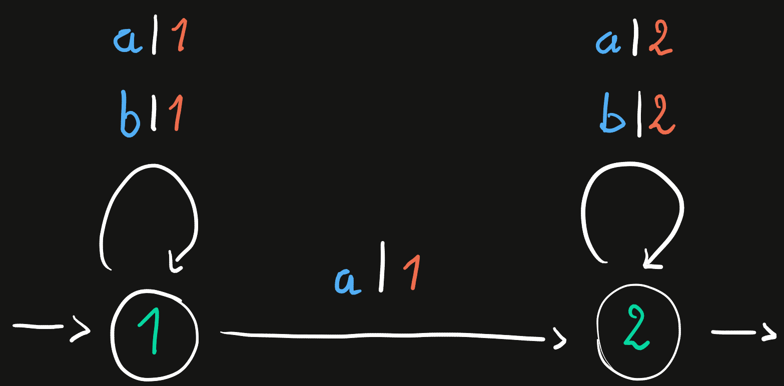

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aab $\longmapsto$ 6

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{cc} 1 & 2 \\ \end{array} \qquad u = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

terminal vector

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{array}{c} 1 \\ 2 \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{array}$$

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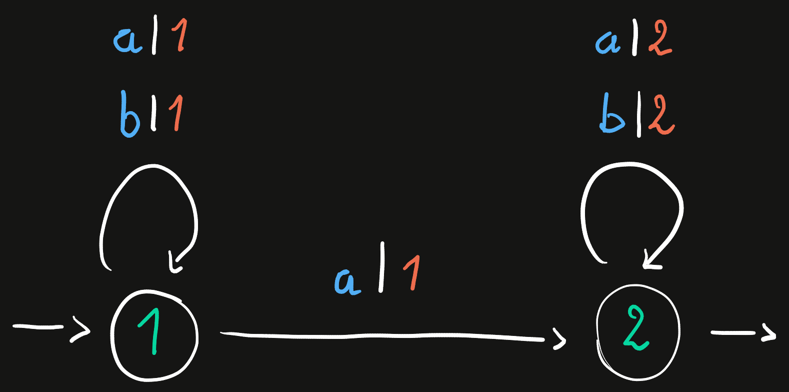

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$aab \longmapsto 6$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{cc} 1 & 2 \\ \end{array}$$
$$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

terminal vector

$$v = \begin{array}{cc} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

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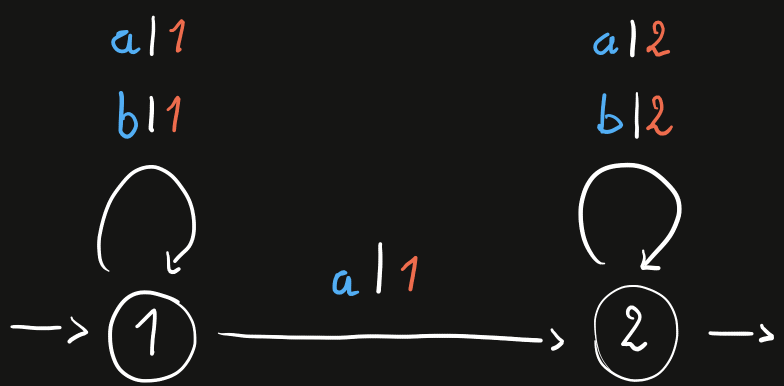
$$aab \longmapsto u \cdot \mu(aab) \cdot v$$
$$= u \cdot \mu(a) \cdot \mu(a) \cdot \mu(b) \cdot v$$
$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= 6$$

Weighted Automata

(WA)



Not always equivalent to a sequential WA

(input deterministic)

Linear representation

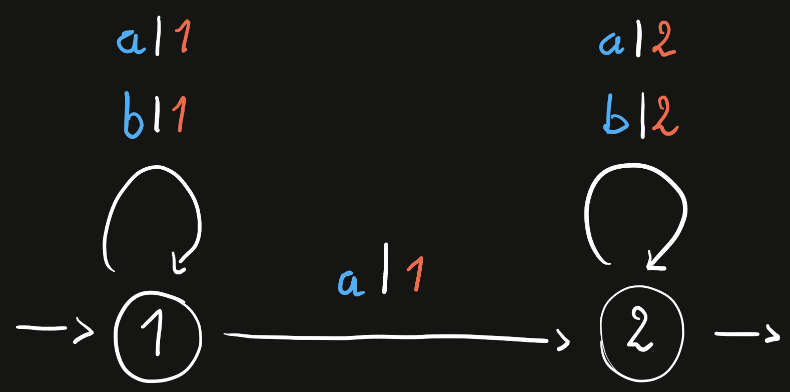
(u, μ, v)

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

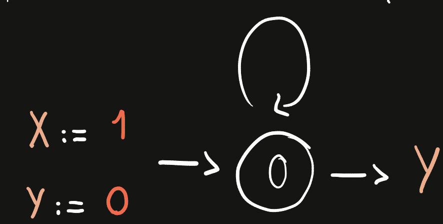
$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

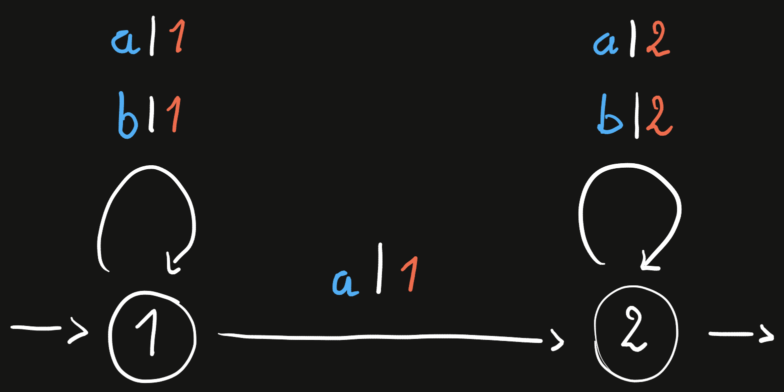
(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$



Weighted Automata

(WA)



Linear representation

(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Cost Register Automata

(CRA) [Alur et al. 2013]

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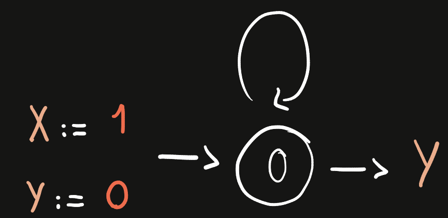


aab :

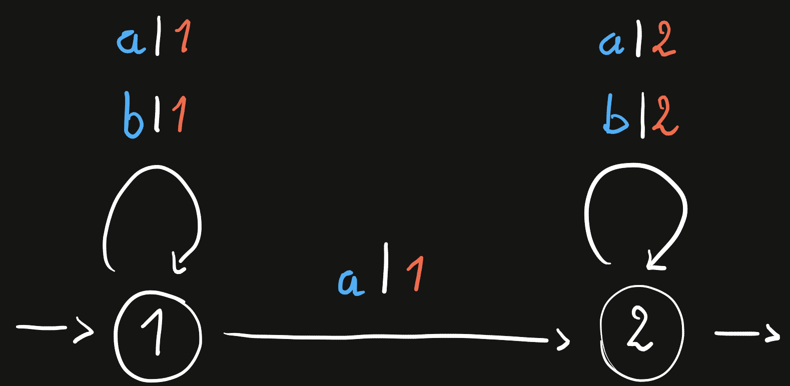
$\to 0$

$X = 1$

$Y = 0$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

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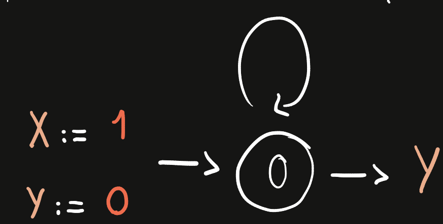


aab :

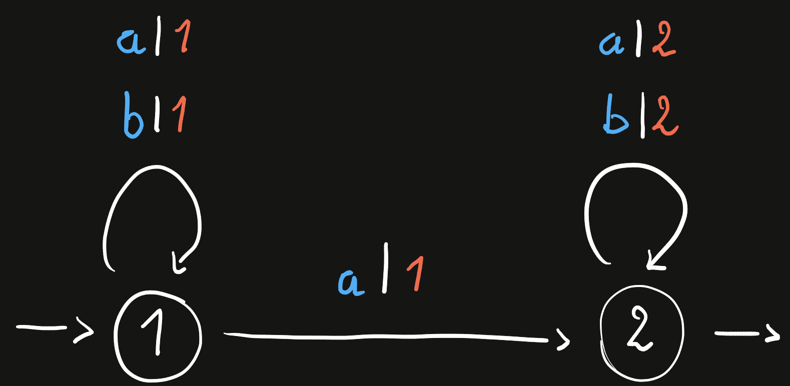
$$\rightarrow \textcircled{0} \xrightarrow{a} \textcircled{0}$$

$X = 1 \longrightarrow 1$

$Y = 0 \longrightarrow 1$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\quad$ $\mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Cost Register Automata

(CRA) $\quad$ [Alur et al. 2013]

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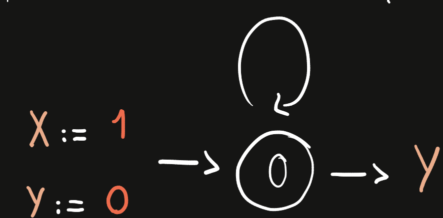


aab :

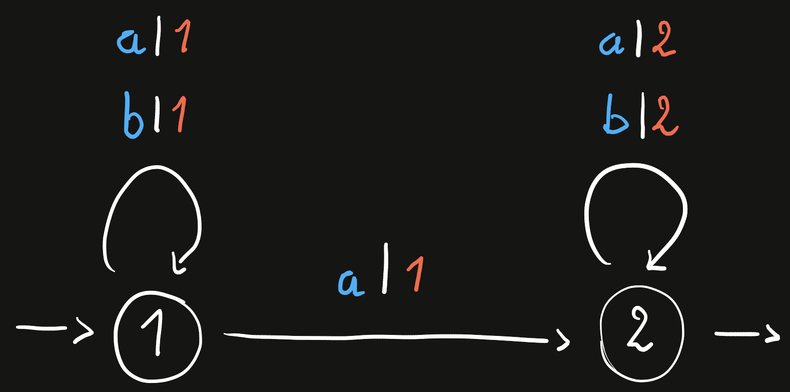
$\to 0 \xrightarrow{a} 0 \xrightarrow{a} 0$

$X = 1 \longrightarrow 1 \longrightarrow 1$

$Y = 0 \longrightarrow 1 \longrightarrow 3$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

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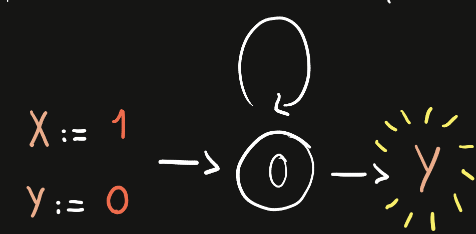


aab :

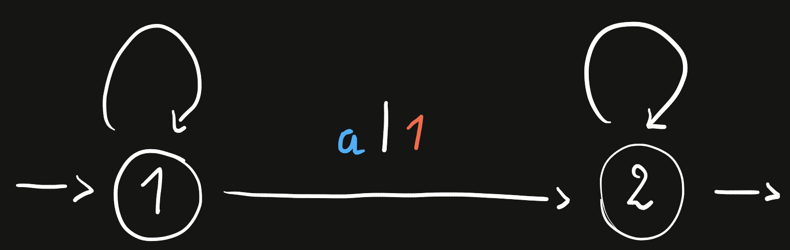
$$\to 0 \xrightarrow{a} 0 \xrightarrow{a} 0 \xrightarrow{b} 0 \to$$

$$X = 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow 1$$

$$Y = 0 \longrightarrow 1 \longrightarrow 3 \longrightarrow 6$$

$$aab \mapsto 6$$

Weighted Automata
(WA)



Linear representation

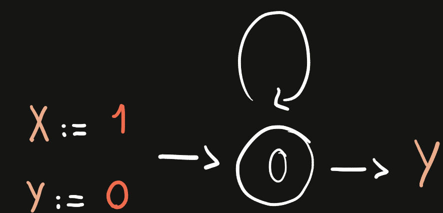
(u, μ, v)

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata
(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \qquad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$

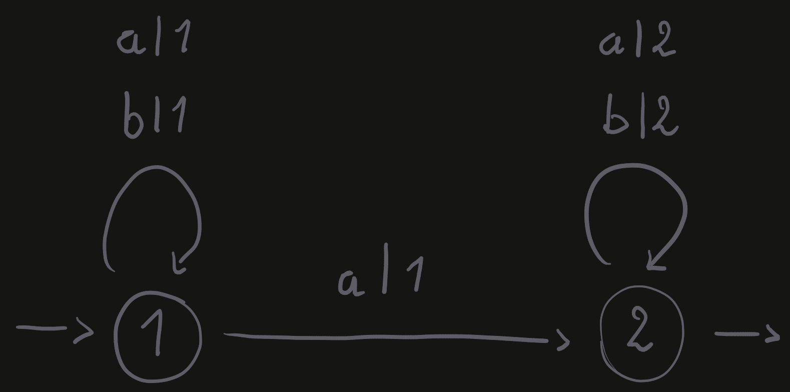


Prop.

- Linear CRA $\iff$ WA
  $(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

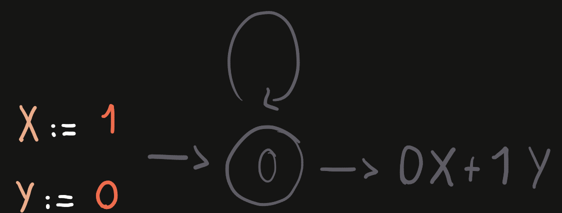
$$u = \begin{array}{cc} X & Y \\ \left(1 \right. & \left. 0 \right) \end{array} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := 1X + 0Y \\ Y := 1X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := 1X + 0Y \\ Y := 0X + 2Y \end{array} \right.$$



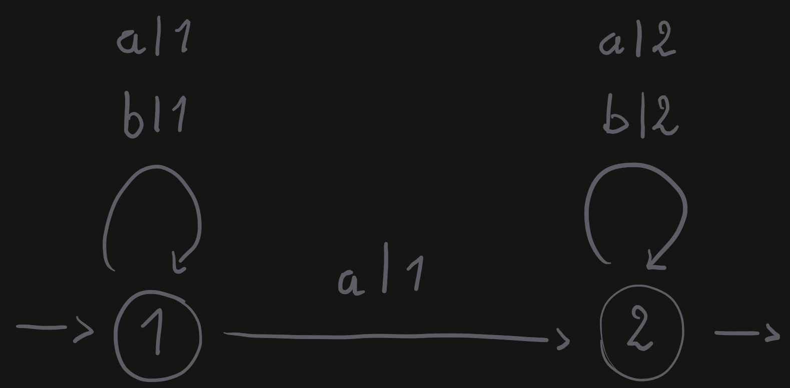
Prop.

- Linear CRA <=> WA

$(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

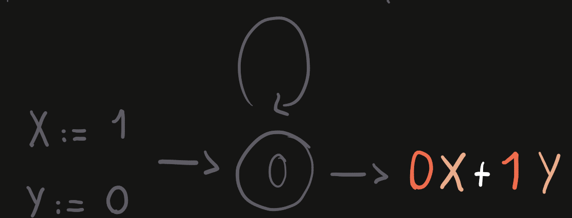
$$u = \begin{array}{c} \;\; X \quad\; Y \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array} \qquad v = \begin{array}{cc} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{array}{c} X \\ Y \end{array} \end{array}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := 1X + 0Y \\ Y := 1X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := 1X + 0Y \\ Y := 0X + 2Y \end{array} \right.$$



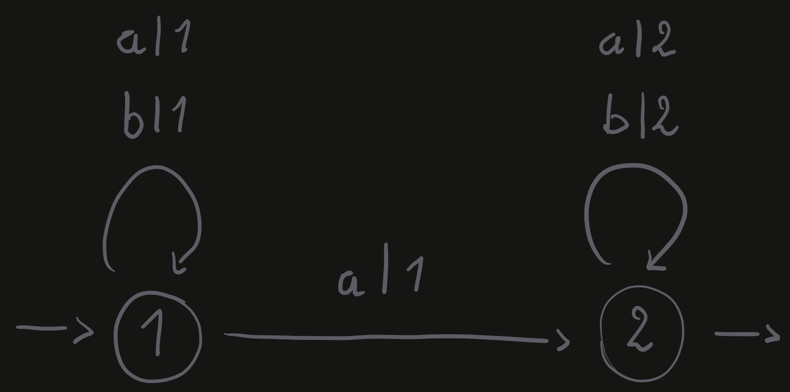
Prop.

- Linear CRA $\Leftrightarrow$ WA

$(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



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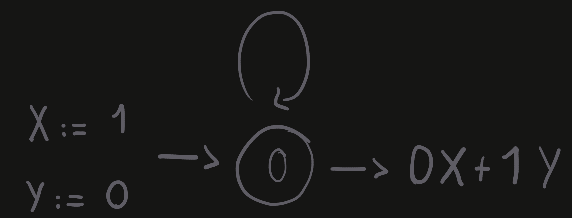

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Weighted Automata

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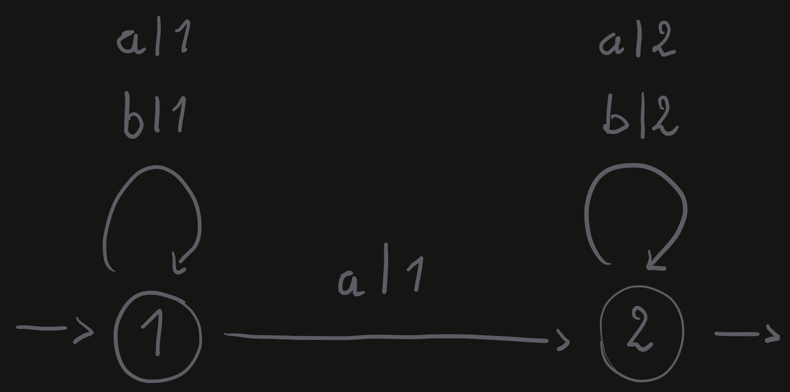

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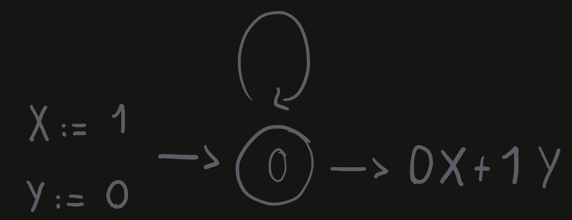

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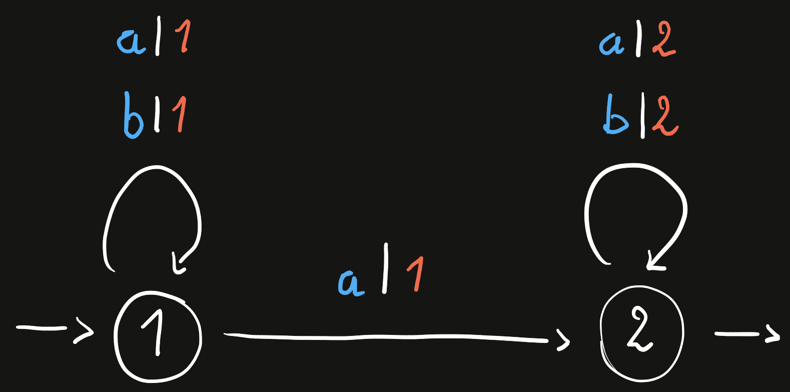

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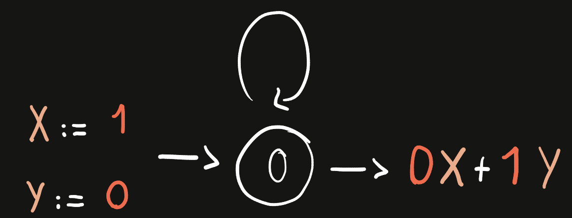

Prop.

- Linear CRA <=> WA

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Weighted Automata

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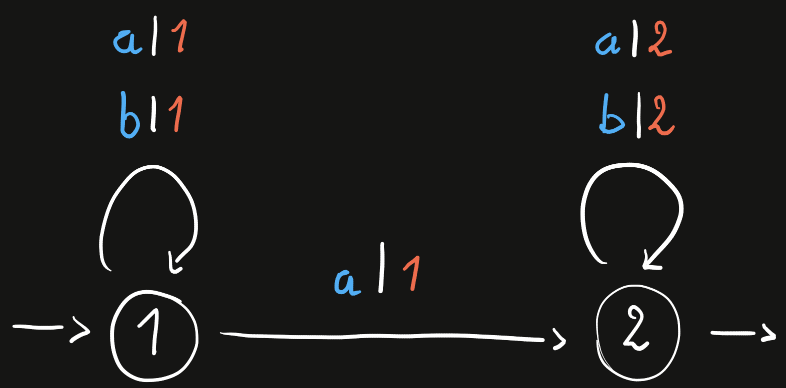

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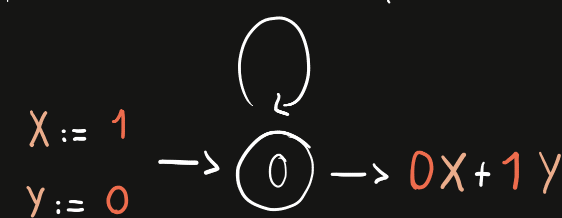

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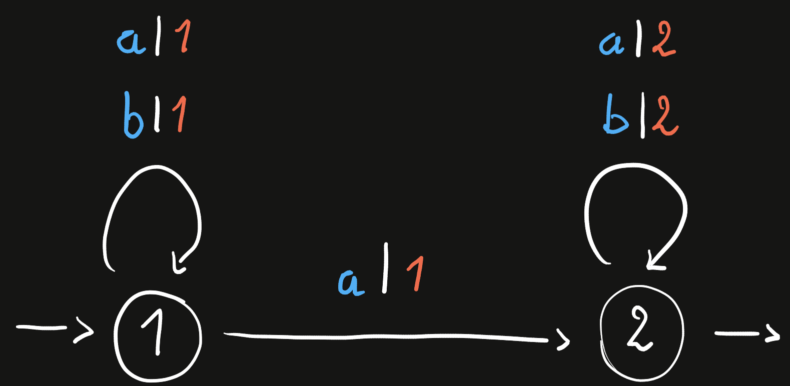

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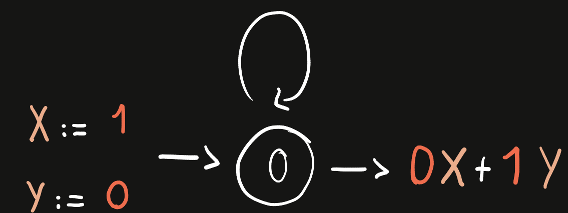

Prop.

- Linear CRA <=> WA
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- 1 Register CRA <=> Sequential WA
  $(X := \alpha X)$

Def. Register complexity of f
= min nb. of registers needed by a CRA to realize f

Register minimization

In: f rational series

$k \in \mathbb{N}$

Q?: $\exists$? CRA realizing f with $\leqslant k$ registers

Register minimization

In: f rational series

$k \in \mathbb{N}$

Q?: $\exists$? CRA realizing f with $\leqslant k$ registers

$\geqslant$

Sequentiality

In: f rational series

Q?: $\exists$? Sequential WA realizing f

Register minimization

In: f rational series

$k \in \mathbb{N}$

Q?: $\exists$? CRA realizing f with $\leq k$ registers

$\geq$

Sequentiality

In: f rational series

Q?: $\exists$? Sequential WA realizing f

[Bell & Smertnig 2023]

Decidable

For WA over a Field

Register minimization

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$k \in \mathbb{N}$

Q?: $\exists$? CRA realizing f with $\leqslant k$ registers

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Sequentiality

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Decidable

For WA over a Field

[Jecker, Mazowiecki, Purser 2024]

PSPACE

For polynomially-ambiguous WA over $\mathbb{Q}$

Register minimization

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Decidable

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[Benalioua, Lhote, Reynier 2024]

2EXPTIME

for WA over $\mathbb{Q}$

[Jecker, Mazowiecki, Purser 2024]

PSPACE

for polynomially-ambiguous WA over $\mathbb{Q}$

Decision problem

$\boxed{\text{Register minimization}}$

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PSPACE

For polynomially-ambiguous WA over $\mathbb{Q}$

Computation problem

Register minimization

In: f rational series

Out: CRA realizing f with the min. nb. of registers

$\geq$

Sequentiality

In: f rational series

Q?: $\exists$? Sequential WA realizing f

[Bell & Smertnig 2023]

Decidable

For WA over a field

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PSPACE

For polynomially-ambiguous WA over $\mathbb{Q}$

Let Σ finite alphabet $\quad \mathcal{R} = (u, \mu, v)$ d-dimensional WA on Σ over $\mathbb{K}$

$\mathbb{K}$ field

Def: Invariant of $\mathcal{R}$

set $\mathcal{I} \subseteq \mathbb{K}^d$ s.t.

- $u \in \mathcal{I}$
- $\forall a \in \Sigma, \forall x \in \mathcal{I}, \; x \cdot \mu(a) \in \mathcal{I}$

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E.g.: $u \cdot \mu(\Sigma^*)$: Reachability set

$\mathbb{K}^d$

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weakest

Def: I is stronger than J

if $I \subseteq J$

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Def: I is stronger than J

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Def: Linear Zariski topology

[Bell & Smertnig 2021]

closed sets : finite unions of vector subspaces of $\mathbb{K}^d$

Z-linear sets

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Z-linear sets

irreducible components

$$S = V_1 \cup V_2 \cup \cdots \cup V_n$$

Length n = nb. of components

Dimension $k = \max\limits_{1 \le i \le n} (\dim(V_i))$

Let Σ finite alphabet, $\mathbb{K}$ field, $\mathcal{R} = (u, \mu, v)$ d-dimensional WA on Σ over $\mathbb{K}$

Def: $\mathbb{Z}$-linear Invariant of $\mathcal{R}$

$\mathbb{Z}$-linear set $I \subseteq \mathbb{K}^d$ s.t.

- $u \in I$
- $\forall a \in \Sigma, \forall x \in I, \; x \cdot \mu(a) \in I$

E.g.: $\overline{u \cdot \mu(\Sigma^*)}^{\ell}$: (Linear Hull) ← strongest

$\mathbb{K}^d$ ← weakest

Def: I is stronger than J if $I \subseteq J$

Def: Linear Zariski topology [Bell & Smertnig 2021]

closed sets: finite unions of vector subspaces of $\mathbb{K}^d$

↑ $\mathbb{Z}$-linear sets

$$S = \underbrace{V_1}_{} \cup V_2 \cup \cdots \cup V_n$$

irreducible components: $V_1, V_2, \dots, V_n$

Length n = nb. of components

Dimension $k = \max_{1 \le i \le n} (\dim(V_i))$

E.g. $\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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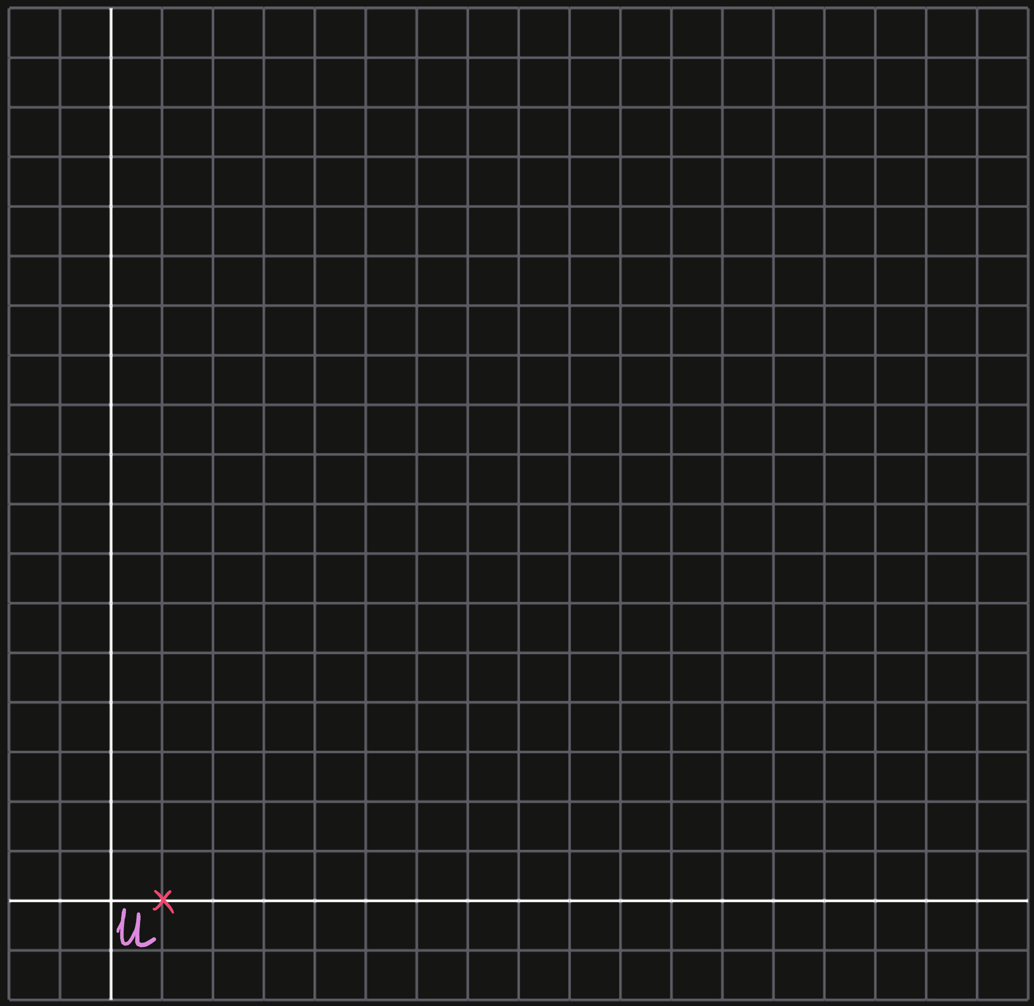

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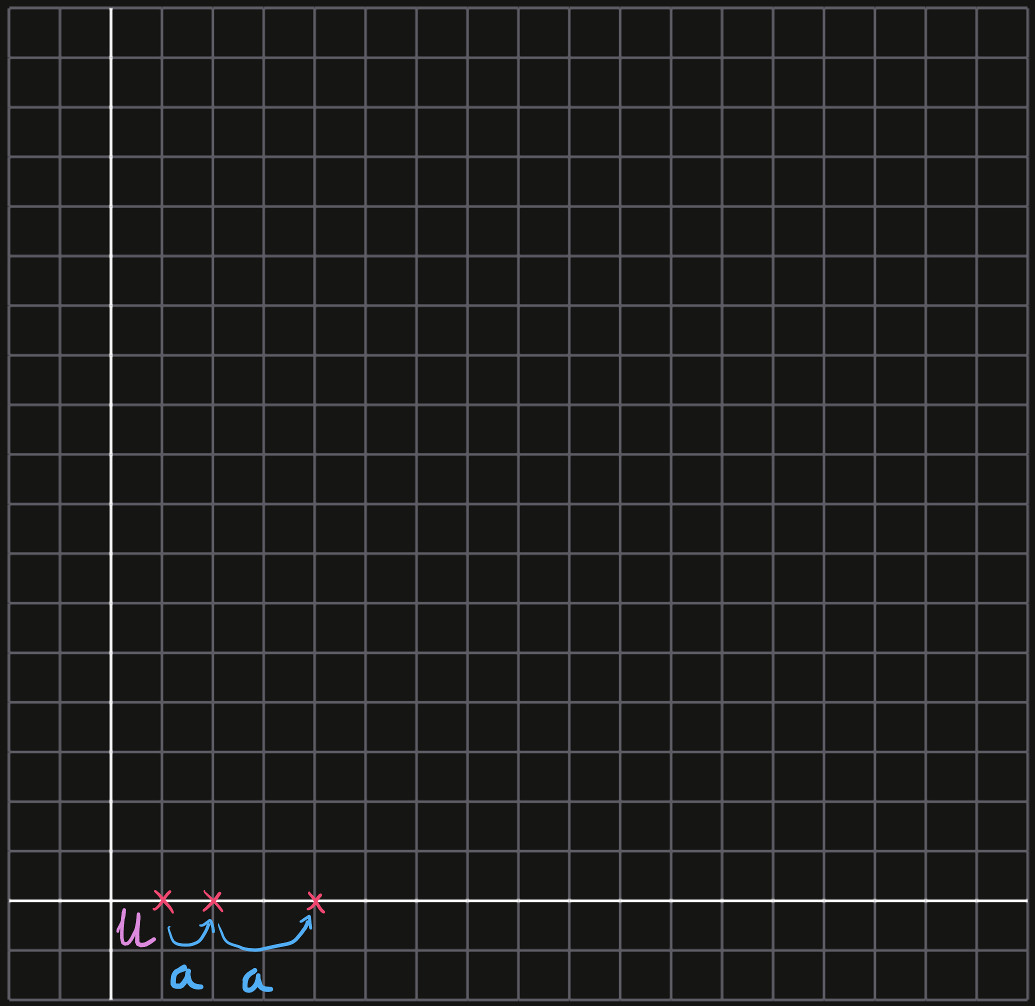

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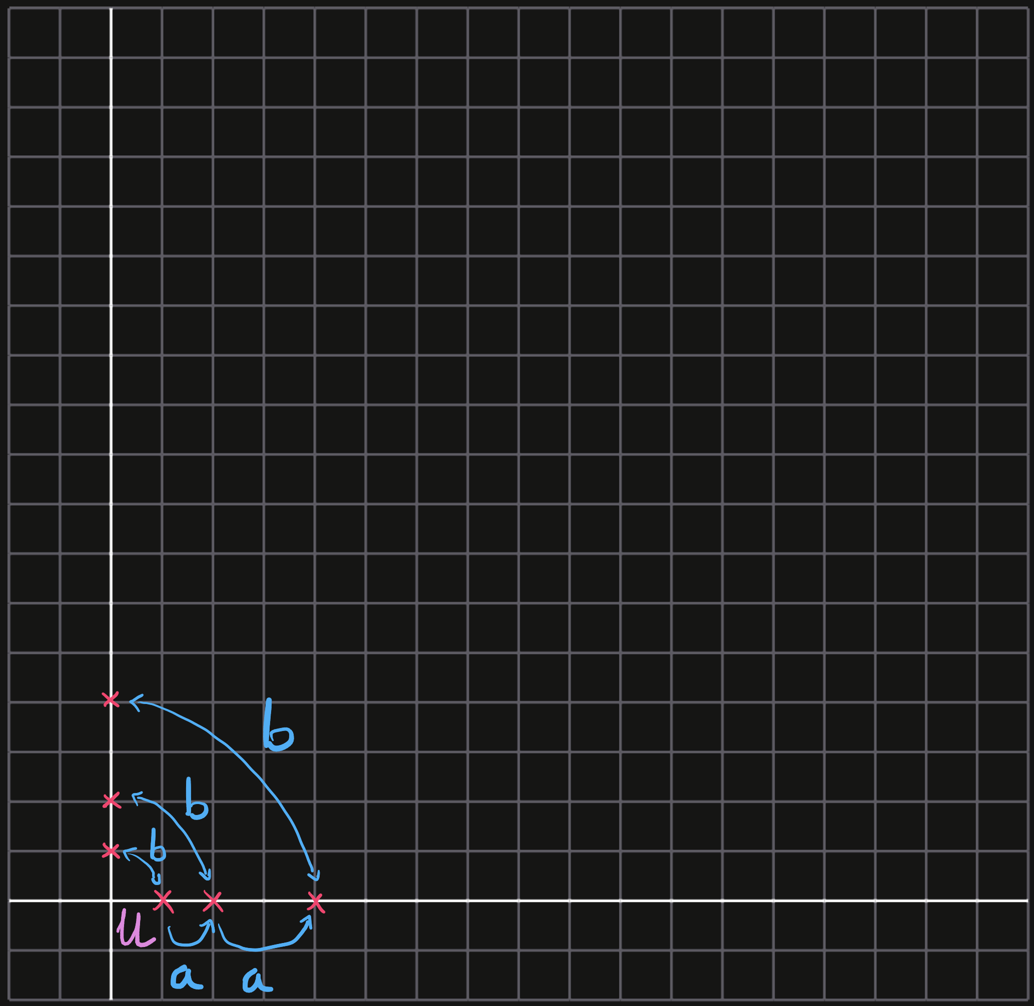

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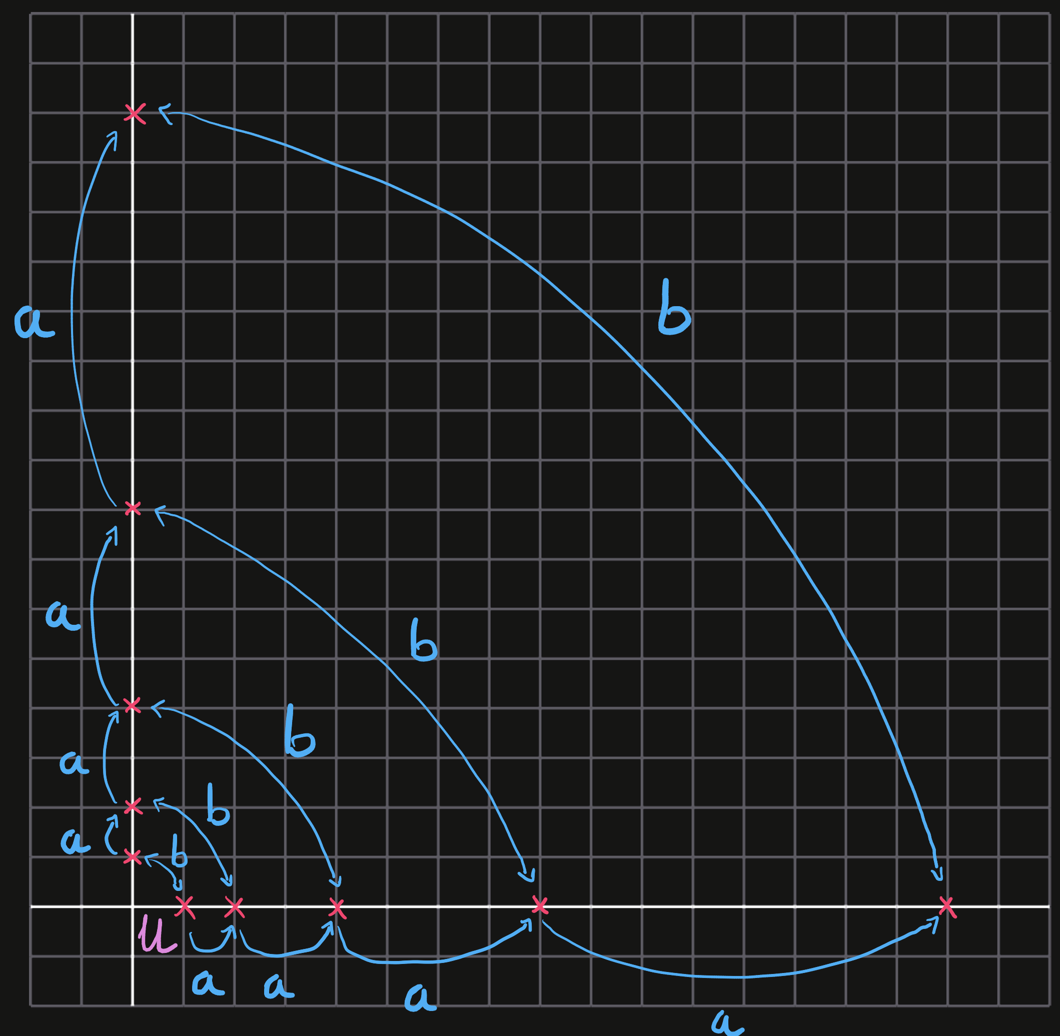

$$w \mapsto \begin{cases} 2^{|w|_a} & \text{if } |w|_b \text{ is even} \\ 0 & \text{else} \end{cases}$$

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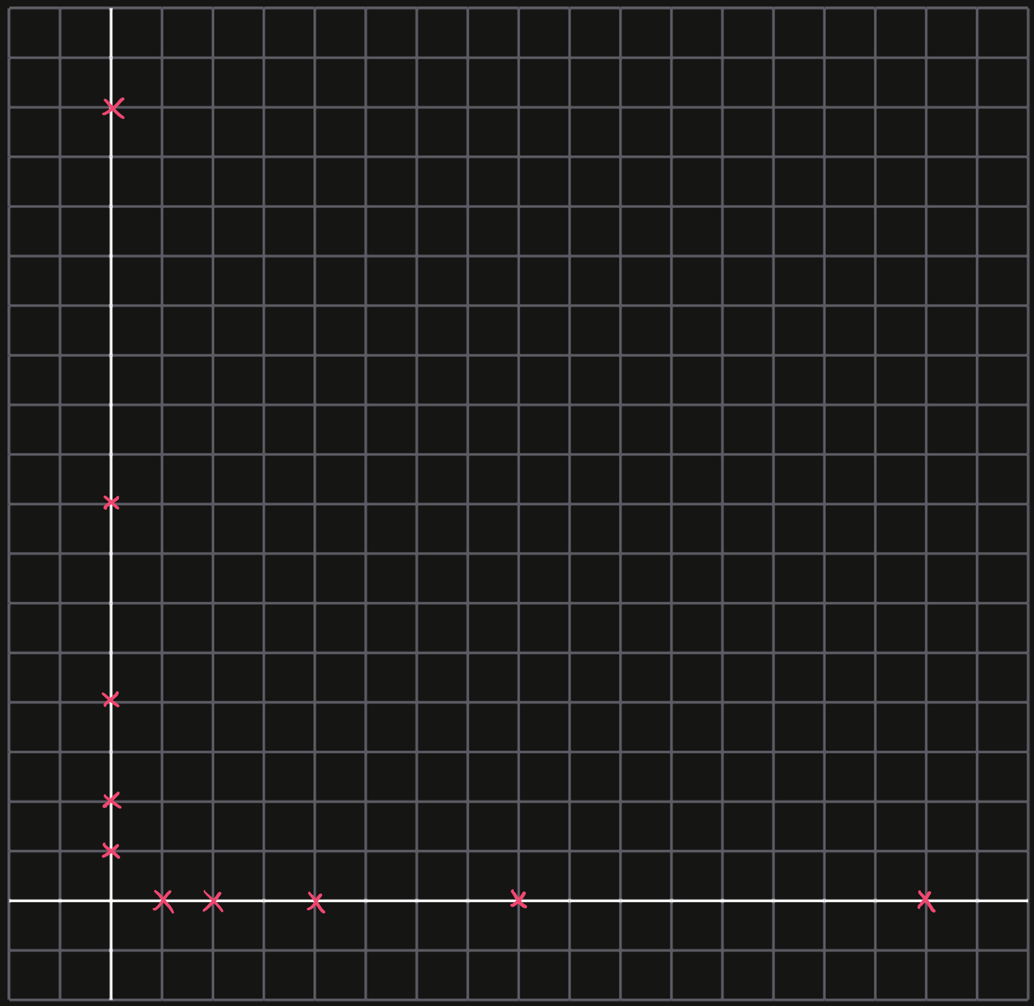

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$$u\mu(\Sigma^*) = \{(2^n \; 0), n \in \mathbb{N}\} \cup \{(0 \; 2^n), n \in \mathbb{N}\}$$

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$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$

Length: 2

Dimension: 1

Let f be a rational series over a Field

[Schützenberger 1961]

→ ∃ minimal WA realizing f

unique up to change of basis

computable in PTIME

[Benalioua, Lhote, Reynier 2024]

Main results

Thm: (Characterization)

∃ CRA for f with n states & k registers

iff

∃ minimal WA $\mathcal{R}$ for f with a

$\mathbb{Z}$-linear invariant I s.t.

$\text{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

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Cor: Register complexity of f

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where (u, μ, v): minimal WA for f

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Thm: over $\mathbb{Q}$, $\overline{u\mu(\Sigma^*)}^{\ell}$ is computable in 2EXPTIME

Let f be a rational series over a field

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Register minimization

is decidable in 2EXPTIME

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Main results

$\boxed{\begin{array}{l}\textbf{Thm: (Characterization)}\\ \exists \text{ CRA for } f \text{ with } n \text{ states } \& \ k \text{ registers}\\ \text{iff}\\ \exists \text{ minimal WA } \mathcal{R} \text{ for } f \text{ with a}\\ \mathbb{Z}\text{-linear invariant } I \text{ s.t.}\\ \mathrm{length}(I) \leqslant n \ \& \ \dim(I) \leqslant k\end{array}}$

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$=$

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[Schützenberger 1961]

→ ∃ minimal WA realizing f

unique up to change of basis

computable in PTIME

[Benalioua, Lhote, Reynier 2024]

Main results

Thm: (Characterization)

∃ CRA for f with n states & k registers

iff

∃ minimal WA $\mathcal{R}$ for f with a $\mathbb{Z}$-linear invariant I s.t.

$\text{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

Cor: Register complexity of f

$=$

$\dim\left(\overline{u\mu(\Sigma^*)}^{\ell}\right)$

where (u,μ,v): minimal WA for f

Register minimization is decidable in 2EXPTIME ←

Thm: over $\mathbb{Q}$, $\overline{u\mu(\Sigma^*)}^{\ell}$ is computable in 2EXPTIME

WA $\to$ CRA

$\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

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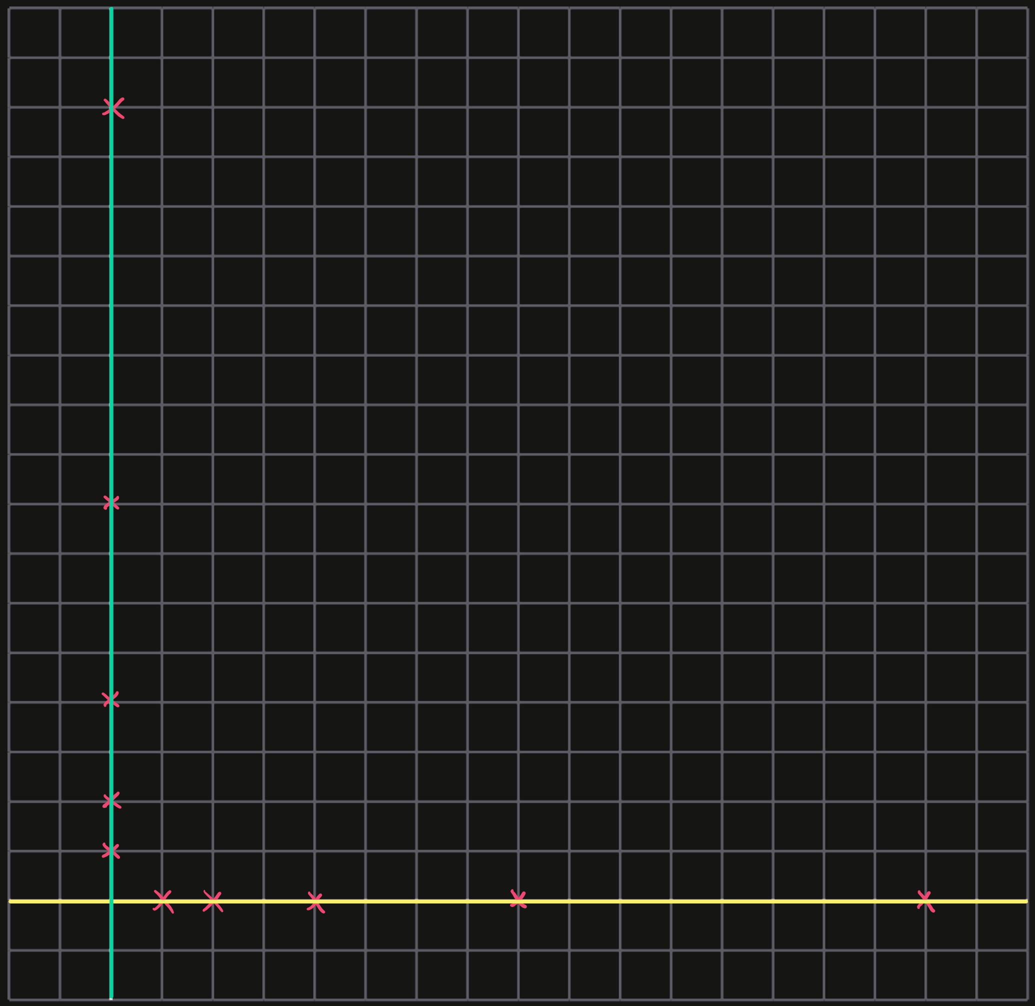

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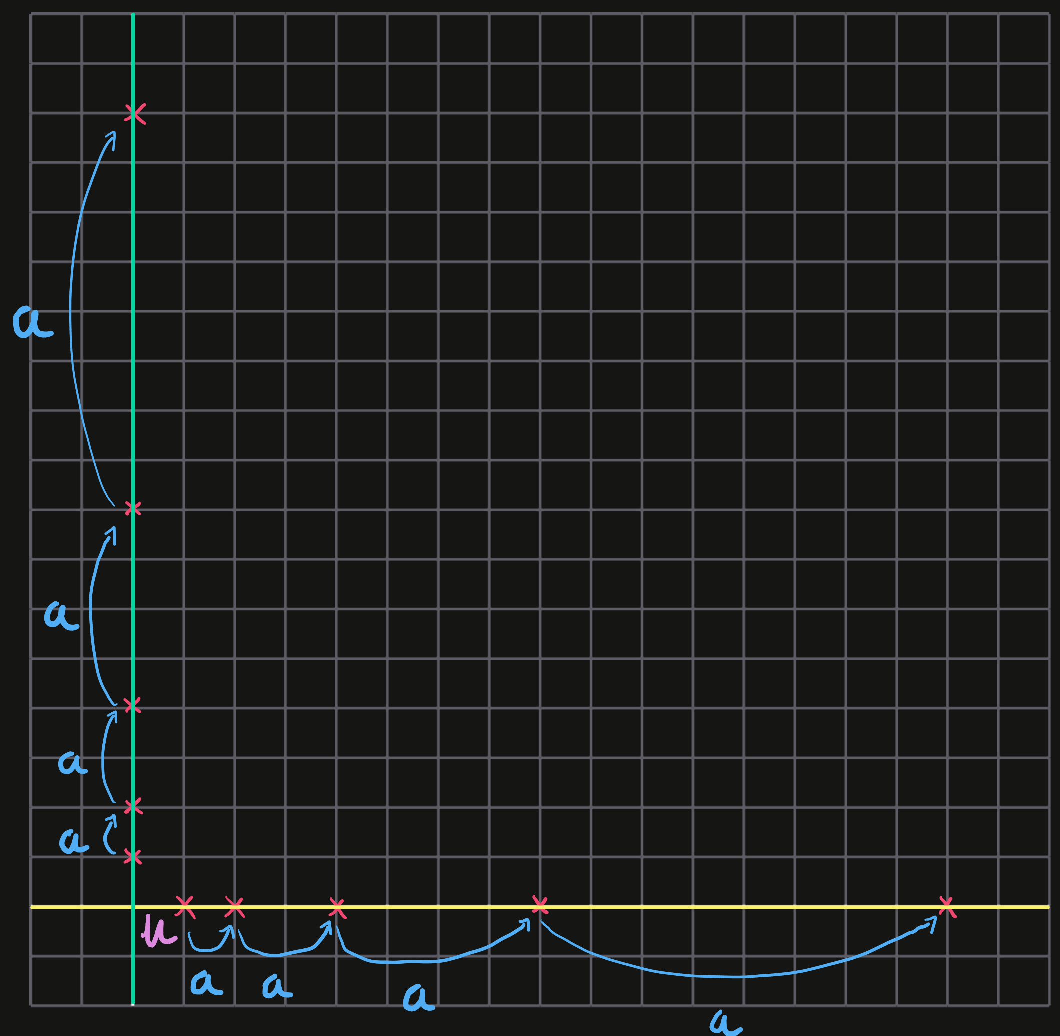

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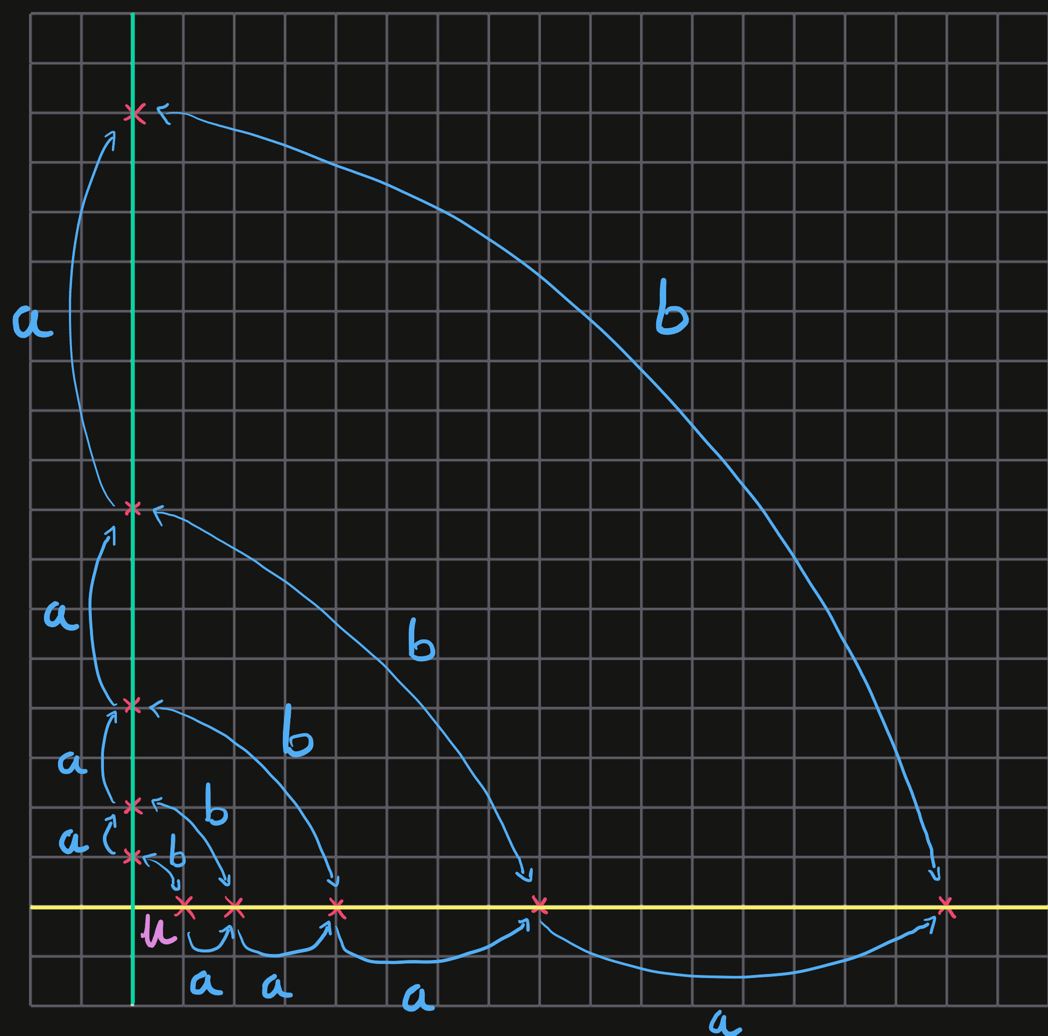

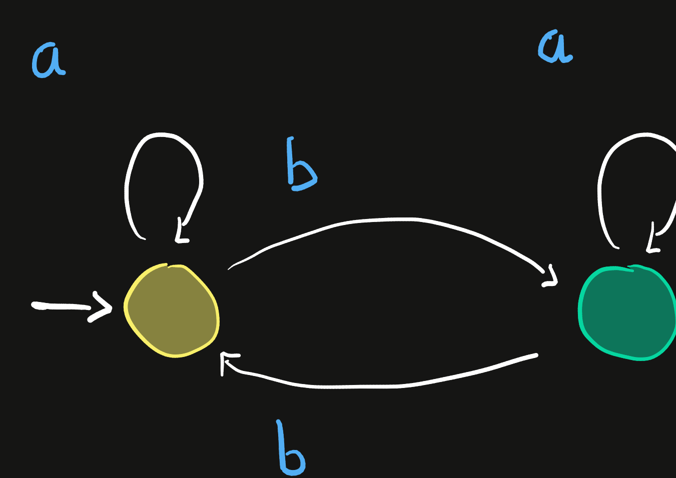

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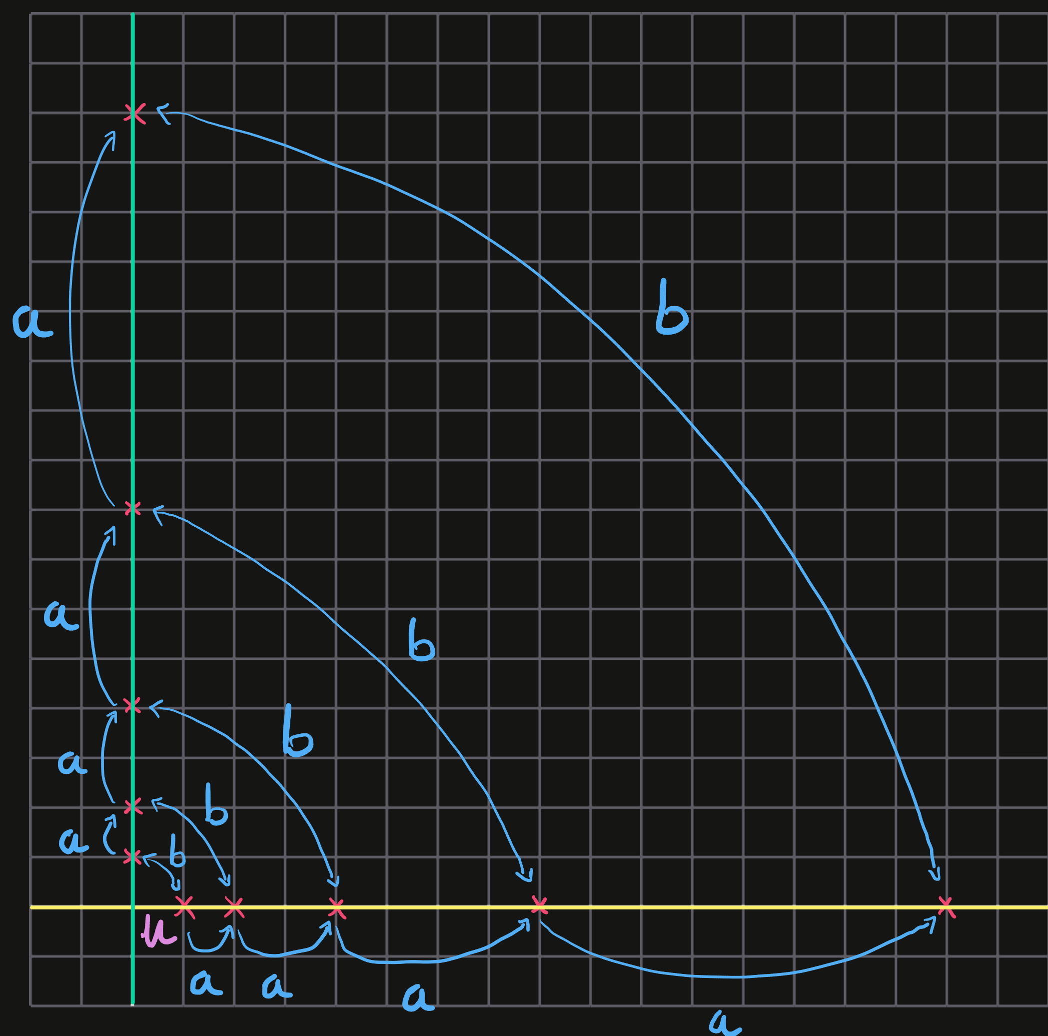

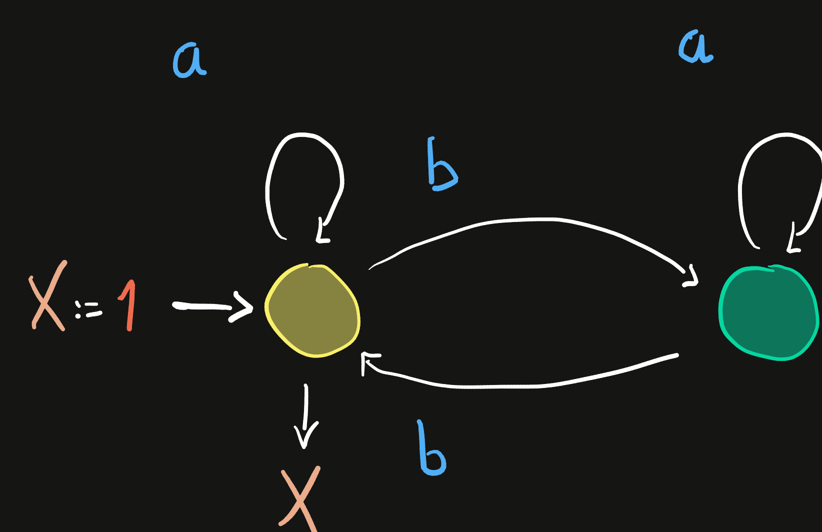

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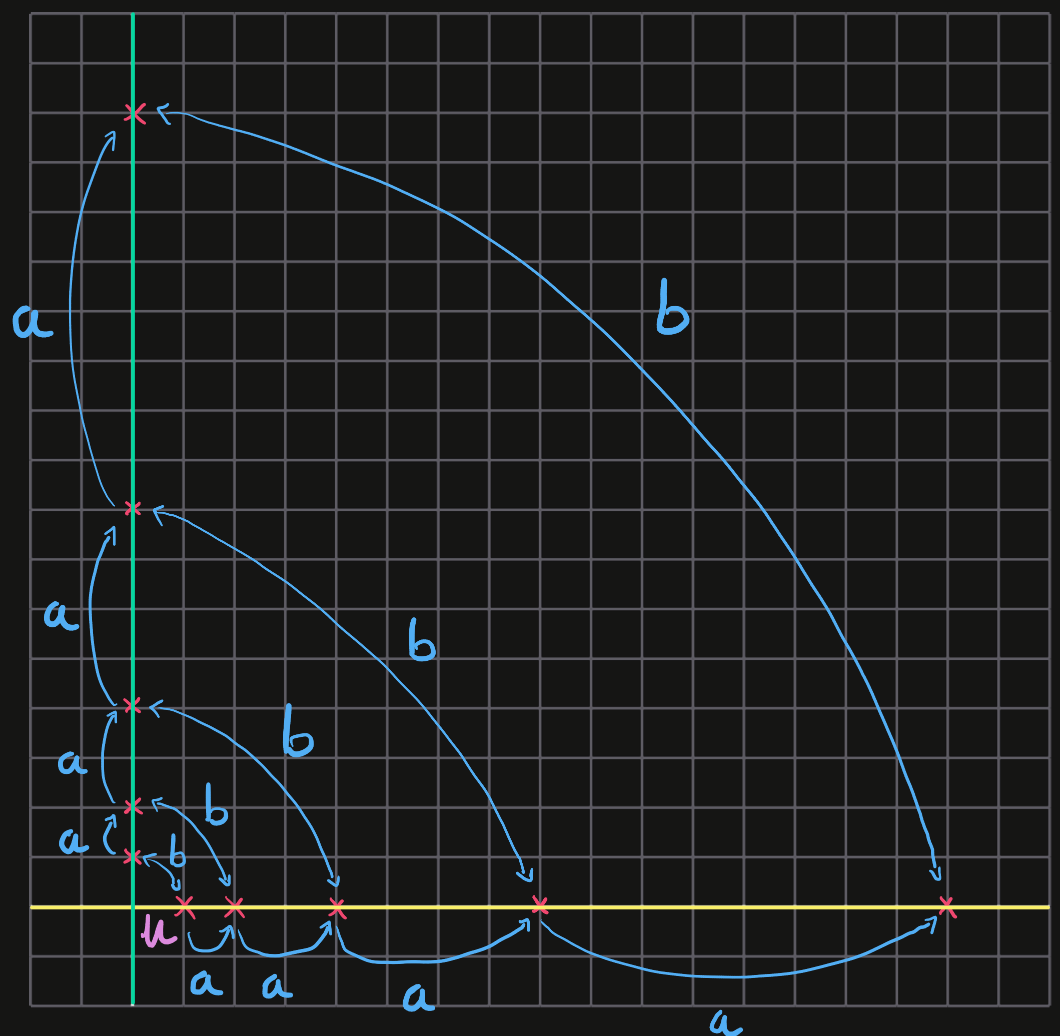

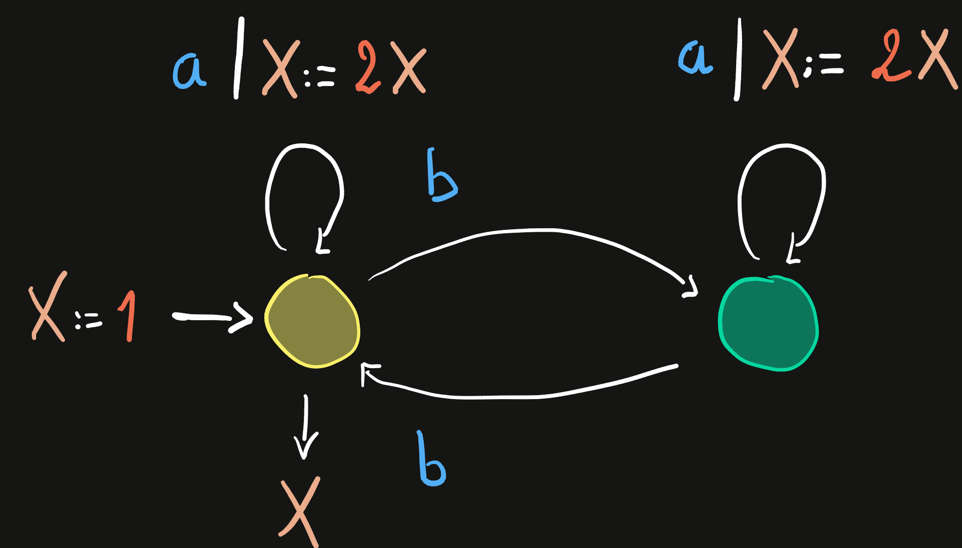

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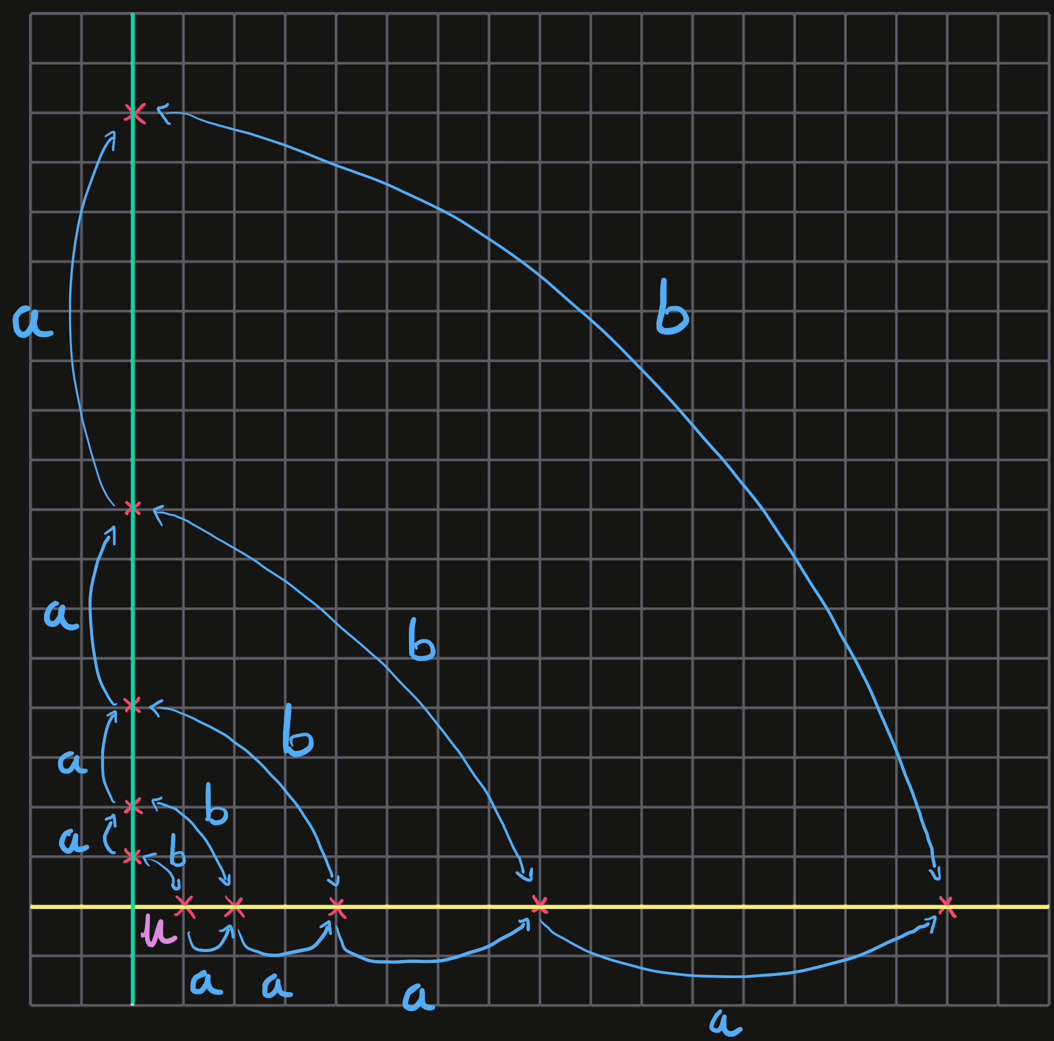

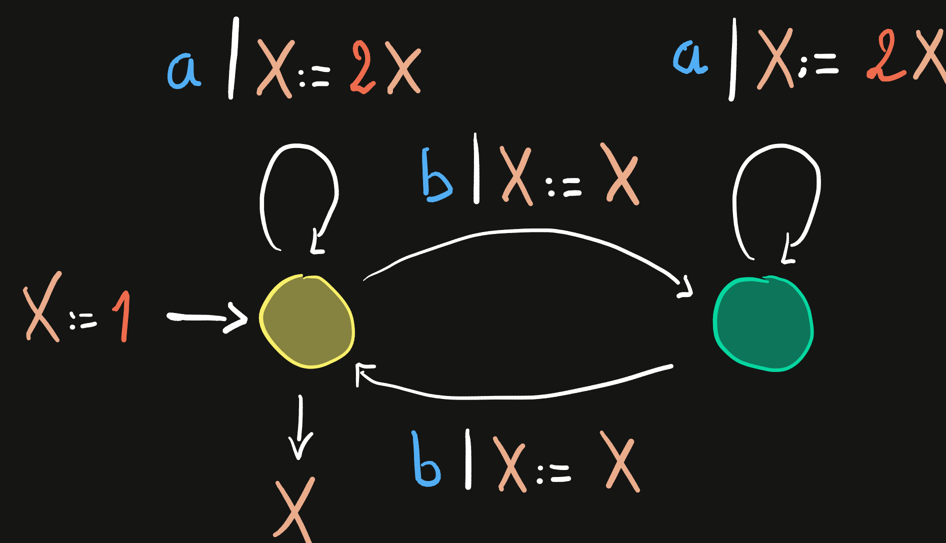

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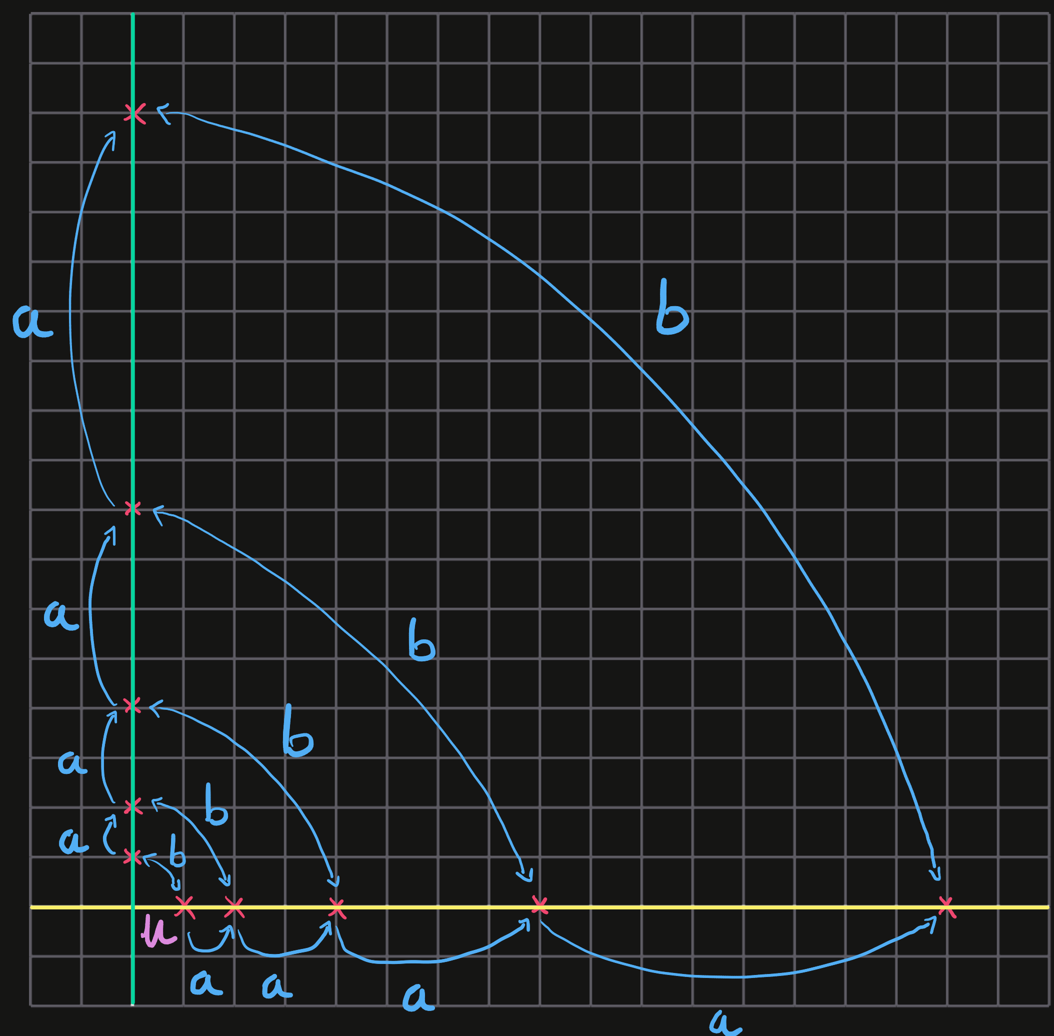

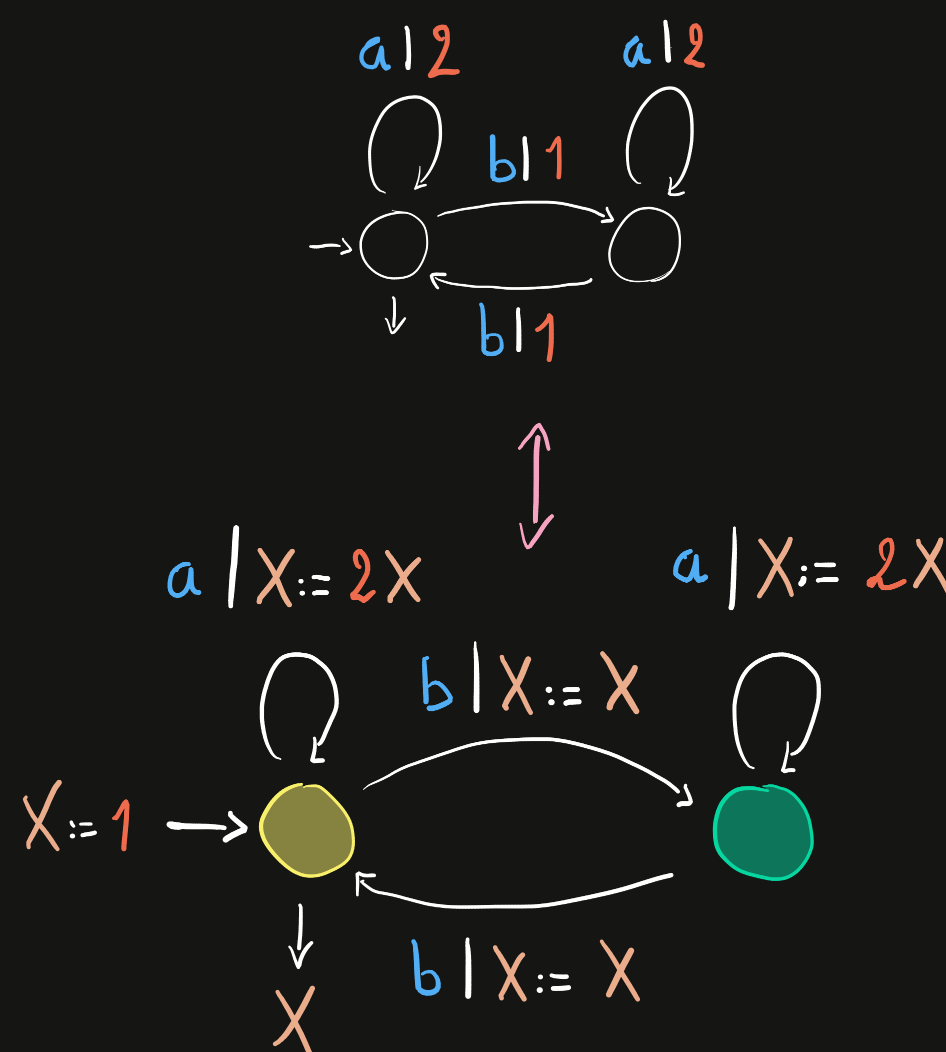

Let f be a rational series over a field

[Schützenberger 1961]

→ ∃ minimal WA realizing f

unique up to change of basis

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Register minimization ←

is decidable in 2EXPTIME

[Benalioua, Lhote, Reynier 2024]

Main results

$\boxed{\begin{array}{l}\text{Thm: (Characterization)}\\ \exists \text{ CRA for } f \text{ with } n \text{ states } \& \ k \text{ registers}\\ \text{iff}\\ \exists \text{ minimal WA } \mathcal{R} \text{ for } f \text{ with a}\\ \text{Z-linear invariant } I \text{ s.t.}\\ \text{length}(I) \leqslant n \ \& \ \dim(I) \leqslant k\end{array}}$

$\boxed{\begin{array}{l}\text{Cor: Register complexity of } f\\ = \dim\left(\overline{u\mu(\Sigma^*)}^{\ell}\right)\\ \text{where } (u,\mu,v)\text{: minimal WA for } f\end{array}}$

$\boxed{\begin{array}{l}\text{Thm: over } \mathbb{Q},\ \overline{u\mu(\Sigma^*)}^{\ell} \text{ is}\\ \text{computable in 2EXPTIME}\end{array}}$

Let $\mathcal{R} = (u, \mu, v)$ be a d-dimensional WA

Prop. given $b \in \mathbb{N}$,

we can compute in time $O(b^{\text{Poly}(d)})$

a $\mathbb{Z}$-lin. invar. I of $\mathcal{R}$ s.t

$\forall$ $\mathbb{Z}$-lin. invar. J of $\mathcal{R}$

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From [Bell & Smertnig 2023]

Prop. let $M_1, \dots, M_r \in GL_d(\mathbb{Q})$

$$\mathrm{length}\left(\overline{\langle M_1, \dots, M_r \rangle}^{\ell} \right) \leqslant 2^{2^{\mathrm{Poly}(d)}}$$

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$\rightsquigarrow$

Prop. let $M_1, \dots, M_r \in \mathbb{Q}^{d \times d}$
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Tightness ?

Prop. let $M_1, \dots, M_r \in \mathbb{Q}^{d \times d}$,

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Ex : For $n \in \mathbb{N}$,

let $M_n = \text{diag}(C_2, C_3, C_5, \dots, C_p)$ where p = largest prime $\leq n$

$C_i = \left(e_i \middle| e_1 \middle| \dots \middle| e_{i-1} \right) \quad \forall i \in \mathbb{N}$

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$M_n \in \mathbb{Q}^{d\times d}$ where $d = \sum_{\substack{p \text{ Prime} \\ p \leq n}} p$

$$\text{length}\left(\overline{\langle M_n \rangle}^{\ell}\right) = |\langle M_n \rangle| = \text{lcm}(2, 3, 5, \dots, p) = \prod_{\substack{p \text{ Prime} \\ p \leq n}} p$$

Conclusion

- Characterization : Z-linear invariant $\longleftrightarrow$ CRA

  $\longrightarrow$ linear hull $\longleftrightarrow$ CRA with min nb. of registers

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  characterization ✓

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Thank you for your attention

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