Rationality of Formal Power Series over Subsemirings

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Formal Power Series

- An alphabet Σ and a semiring R.
- A formal power series is

$$f : \Sigma^* \to R.$$

- Equivalently,

$$f = \sum_{w \in \Sigma^*} f(w) \cdot w.$$

Definition (Rational Formal Power Series)
A series is rational if it lies in the rational closure (closed under sum, product, and Kleene star) of the polynomials $R\langle\Sigma\rangle$ (finite-support series).

Weighted automaton

A weighted automaton over $\mathbb{R}$ is

$$\mathcal{A} = (\Sigma, Q, \alpha \in \mathbb{R}^Q, (\Delta(a) \in \mathbb{R}^{Q \times Q})_{a \in \Sigma}, \eta \in \mathbb{R}^Q).$$

- Q: finite set of states
- α: initial weight vector
- $\Delta(a)$: transition matrix for letter a
- η: final weight vector

It recognizes the series

$$f(w) = \alpha^T \Delta(w) \eta, \Delta(w) = \Delta(a_1)\Delta(a_2) \cdots \Delta(a_n), w = a_1 a_2 \cdots a_n.$$

Definition (Recognizable Series)
A series recognized by a weighted automaton is called recognizable.

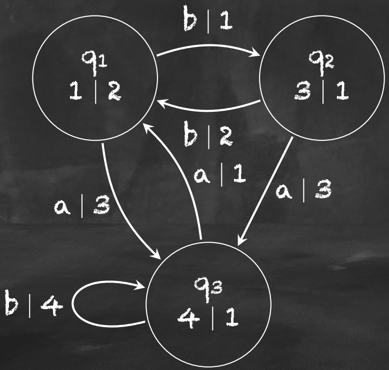
Kleene-Schützenberger Theorem

- Rational series generalize regular languages to the weighted setting.
- Kleene's theorem: regular languages are those recognized by finite automaton.

Theorem (Schützenberger, 1961)
A formal power series is recognizable if and only if it is rational.

Weighted automaton Example

(Example from Balle-Mohri, Theoretical Computer Science, Vol. 716 (2018))



$$\alpha = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \quad \eta = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\Delta(a) = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Delta(b) = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$f(w) = \alpha^T \Delta(w) \eta$$

$$f(\varepsilon) = 9, \; f(a) = 20$$

Rationality over Fields

- Suppose R is a field.
- When is a series $f : \Sigma^* \to R$ rational?

Definition (Hankel Matrix of a Power Series)
The Hankel matrix H_f of f is the (bi-infinite) matrix indexed by $(u, v) \in \Sigma^* \times \Sigma^*$, with

$$H_f(u, v) \;=\; f(uv).$$

Theorem (Fliess' Theorem)
If R is a field, the size of the minimal automaton recognizing f equals $\mathrm{rank}(H_f)$.

Fliess' Theorem: Proof Sketch ($\Rightarrow$)

$\mathcal{A} = (Q, \alpha, \{\Delta(a)\}, \eta)$ recognizes f.

- Define $P \in \mathbb{R}^{\Sigma^* \times Q}$ and $S \in \mathbb{R}^{Q \times \Sigma^*}$ by

$$P(u, q) := q\text{-th entry of } (\alpha^T \Delta(u)),$$

$$S(q, v) := q\text{-th entry of } (\Delta(v)\,\eta).$$

- Then

$$H_f(u, v) = f(uv) = \sum_{q \in Q} P(u, q)\, S(q, v).$$

- Hence

$$H_f = PS \implies \operatorname{rank}(H_f) \leqslant |Q|.$$

Fliess' Theorem: Proof Sketch ($\Leftarrow$)

- Since $\mathrm{rank}(H_f)$ is finite, choose a set $X \subseteq \Sigma^*$ indexing a basis of rows of H_f (w.l.o.g. $\varepsilon \in X$).

Define the automaton $\mathcal{A} = (X, \alpha, \{\Delta(a)\}, \eta)$ recognizing f by:

- States: X.
- Initial vector: $\alpha = e_1$.
- Final weights: $\eta = H_f(X, \varepsilon)$.
- Transitions $\Delta(a)$ satisfying

$$H_f(Xa, \Sigma^*) \; = \; \Delta(a)\left(H_f(X, \Sigma^*)\right).$$

Proof completes by induction.

Rationality over Subsemirings

- Given a weighted automaton over $\mathbb{R}$ recognizing f.
- A subsemiring $S \subseteq \mathbb{R}$.
- Assume $f(\Sigma^*) \subseteq S$.

Problem (Rationality over S)

Is f rational over S? Equivalently, does there exist a weighted automaton over S that recognizes f?

Focus: $R = \mathbb{R}$ and $S = \mathbb{R}_{\geqslant 0}$

Problem (Non-negative Weights)

Given a weighted automaton over $\mathbb{R}$ with $f(\Sigma^*) \subseteq \mathbb{R}_{\geqslant 0}$, decide if f can be recognized by a weighted automaton with non-negative weights.

- Is there an analogue of Fliess' theorem over $\mathbb{R}_{\geqslant 0}$?
- How do we define $\mathrm{rank}(H_f)$ over $\mathbb{R}_{\geqslant 0}$?

Definition (Minimum Size)

For $f(\Sigma^*) \subseteq S$ rational over S, define

$$\tau_S(f) := \text{size of the smallest automaton for } f.$$

Non-negative Rank

Definition (Non-negative Rank)

For $A \in \mathbb{R}_{\geqslant 0}^{\Sigma^* \times \Sigma^*}$, the non-negative rank $\mathrm{rank}_+(A)$ is the smallest $q \in \mathbb{N}$ such that

$$A = BC, \quad B \in \mathbb{R}_{\geqslant 0}^{\Sigma^* \times q}, \quad C \in \mathbb{R}_{\geqslant 0}^{q \times \Sigma^*}.$$

$$\mathrm{rank}(A) \leqslant \mathrm{rank}_+(A).$$

Lemma

For $f : \Sigma^* \to \mathbb{R}$ with $f(\Sigma^*) \subseteq \mathbb{R}_{\geqslant 0}$,

$$\mathrm{rank}_+(H_f) \leqslant \tau_{\mathbb{R}_{\geqslant 0}}(f) = \tau_+(f).$$

Residual Non-negative Rank

Definition (Residual Non-negative Rank)

For $A \in \mathbb{R}_{\geqslant 0}^{\Sigma^* \times \Sigma^*}$, the residual non-negative rank $\mathrm{rrank}_+(A)$ is the smallest $q \in \mathbb{N}$ such that

$$A = BC, \quad B \in \mathbb{R}_{\geqslant 0}^{\Sigma^* \times q}, \quad C \in \mathbb{R}_{\geqslant 0}^{q \times \Sigma^*},$$

where each row of C is a row of A.

$$\mathrm{rank}(A) \leqslant \mathrm{rank}_+(A) \leqslant \mathrm{rrank}_+(A).$$

Lemma

For $f : \Sigma^* \to \mathbb{R}$ with $f(\Sigma^*) \subseteq \mathbb{R}_{\geqslant 0}$,

$$\tau_{\mathbb{R}_{\geqslant 0}}(f) = \tau_+(f) \leqslant \mathrm{rrank}_+(H_f).$$

Fliess' Theorem over $\mathbb{R}_{\geqslant 0}$

Theorem (Fliess' Theorem)
For $f : \Sigma^* \to \mathbb{R}$ with $f(\Sigma^*) \subseteq \mathbb{R}_{\geqslant 0}$,

$$\mathrm{rank}_+(H_f) \leqslant \tau_{\mathbb{R}_{\geqslant 0}}(f) = \tau_+(f) \leqslant \mathrm{rrank}_+(H_f).$$

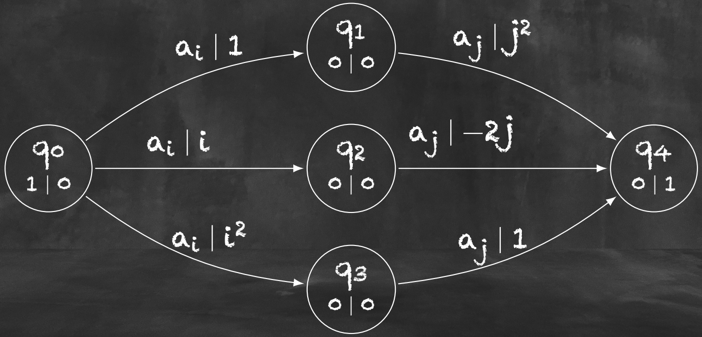
- Define $\tau(f) := \tau_{\mathbb{R}}(f)$.

Question
If $\tau_+(f)$ is finite, is there an explicit bound in terms of $\tau(f)$ and $|\Sigma|$?

Comparing $\tau(f)$ and $\tau_+(f)$

- Let $\Sigma = \{a_1, \ldots, a_n\}$.
- Define f by

$$f(a_i a_j) := (i-j)^2, \quad f(w) = 0 \text{ otherwise.}$$



- $\tau_+(f) \geqslant \mathrm{rank}_+(H_f) = \Theta(\log n)$.

$\mathbb{R}_{\geqslant 0}$-Rationality in One Variable

Theorem (Soittola, 1976)
If a rational series with non-negative coefficients has a dominating eigenvalue, then it is $\mathbb{R}_{\geqslant 0}$-rational.

Theorem (Characterization)
A series over $\mathbb{R}_{\geqslant 0}$ is $\mathbb{R}_{\geqslant 0}$-rational if and only if it is the merge of polynomials and rational series having a dominating eigenvalue.

This characterization follows from Soittola's theorem and Perron-Frobenius theory.

Conjecture on $\mathbb{R}_{\geqslant 0}$-Rationality

Conjecture

Let $f : \Sigma^* \to \mathbb{R}$ be rational. Then f is not $\mathbb{R}_{\geqslant 0}$-rational if and only if there exists a word w such that the sequence $n \mapsto f(w^n)$ is not $\mathbb{R}_{\geqslant 0}$-rational.

- If the weighted automaton can be taken over $\mathbb{Q}_{\geqslant 0}$, then this conjecture would imply decidability of our problem.
- In parallel, search for:
  - A word w that falsifies $\mathbb{R}_{\geqslant 0}$-rationality.
  - A weighted automaton over $\mathbb{Q}_{\geqslant 0}$ recognizing f.

Counterexample

Theorem
The conjecture is false.

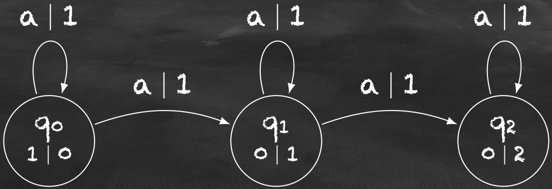
For $\Sigma = \{a, b\}$, define

$$S(w) := \left(|w|_a - |w|_b\right)^2.$$

- S is rational but not $\mathbb{R}_{\geqslant 0}$-rational.
- For each w,

$$S_w(n) = S(w^n) = \left(|w|_a - |w|_b\right)^2 n^2$$

is $\mathbb{R}_{\geqslant 0}$-rational.



Conclusion

- Non-negative and residual ranks bound automaton size.
- Rationality in one variable is decidable (Soittola-Perron-Frobenius).
- Multivariate case over $\mathbb{R}_{\geqslant 0}$: still open and difficult.
- Many open problems remain.