The Membership Problem for Hypergeometric Sequences

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The Membership Problem for recurrence sequences

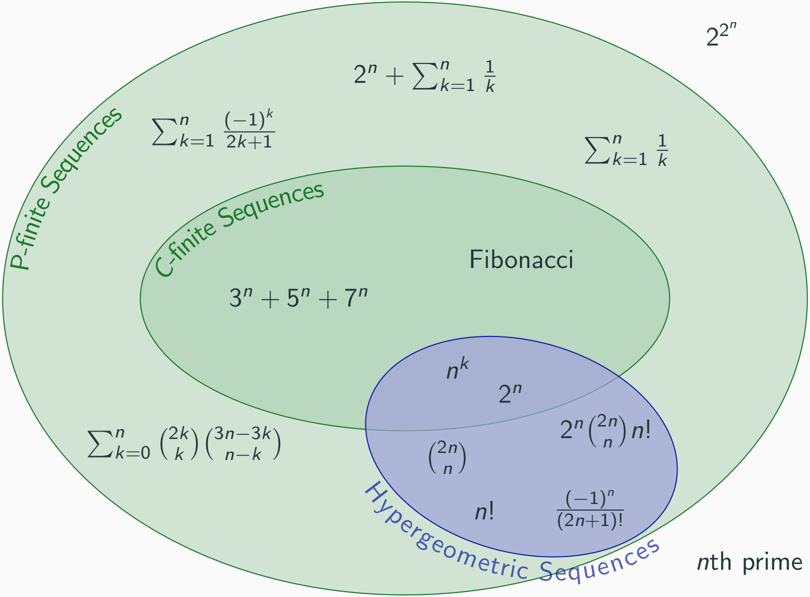
MP: Given a recurrence sequence $\langle u_n \rangle_{n=0}^{\infty}$ and target t, procedurally determine whether $\exists n \in \mathbb{N}$ st $u_n = t$.

For C-finite sequences, Membership (Skolem) is an open problem.

> *"It is faintly outrageous that this problem is still open... we do not know how to decide the Halting Problem even for 'linear' automata!"*
> —*Tao (2008)*

> *"a mathematical embarrassment"*
> —*Lipton (2010)*

A landscape of recurrence sequences



What is a hypergeometric sequence?

Sequence $\langle u_n \rangle_{n=0}^{\infty} \subset \mathbb{Q}$ is *hypergeometric* if it satisfies a first-order recurrence relation with polynomial coefficients, i.e.,

$$f(n)u_n = g(n)u_{n-1} \qquad \text{where } f(x), g(x) \in \mathbb{Z}[x].$$

$$u_0$$

$$u_1 = \frac{g(1)}{f(1)} u_0$$

$$u_2 = \frac{g(2)g(1)}{f(2)f(1)} u_0$$

$$\vdots$$

$$u_n = \left(\prod_{k=1}^{n} \frac{g(k)}{f(k)} \right) u_0$$

If $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are hypergeometric then so are:

1. $\langle u_n v_n \rangle_n$
2. $\langle 1/u_n \rangle_n$ ($u_n \neq 0 \ \forall n$)
3. $\langle u_{pn+q} \rangle_n$ (fixed $p, q \in \mathbb{N}$)

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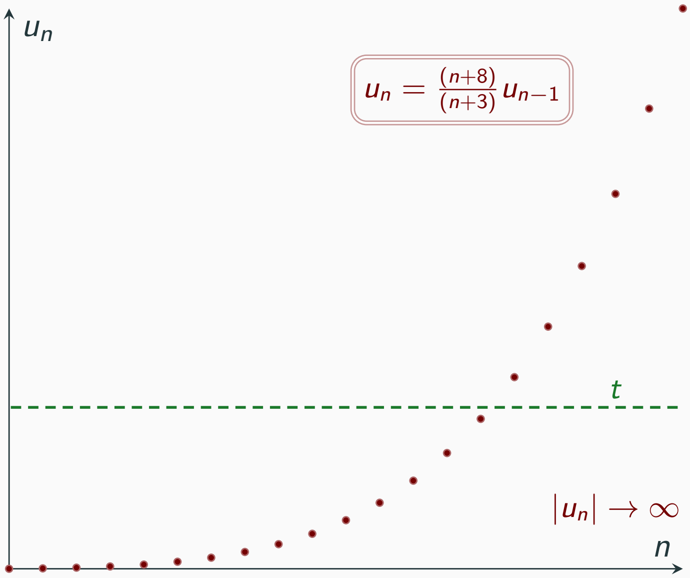
We can detect when $|u_n| \to \infty$. MP is decidable for such seqs.



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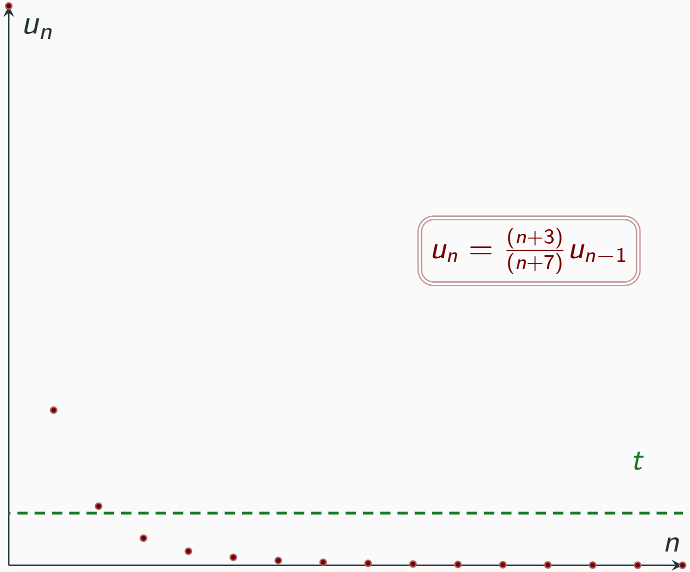
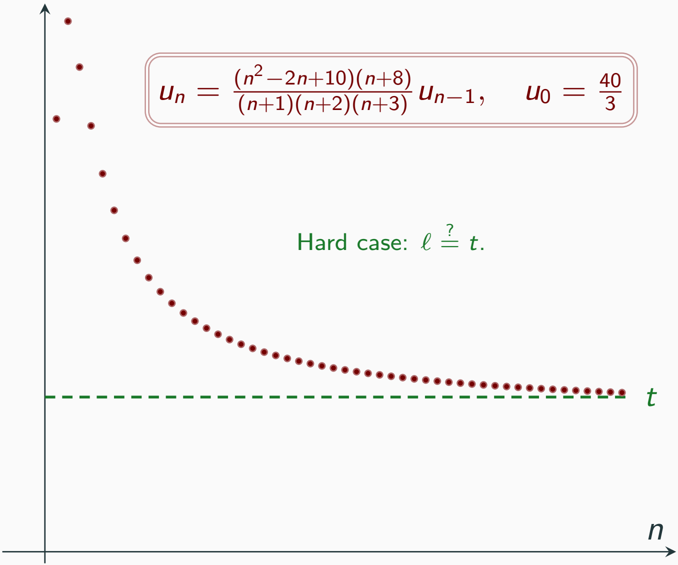


When $u_n \to \ell$ ($\ell \neq 0$), there is an obstacle to MP.



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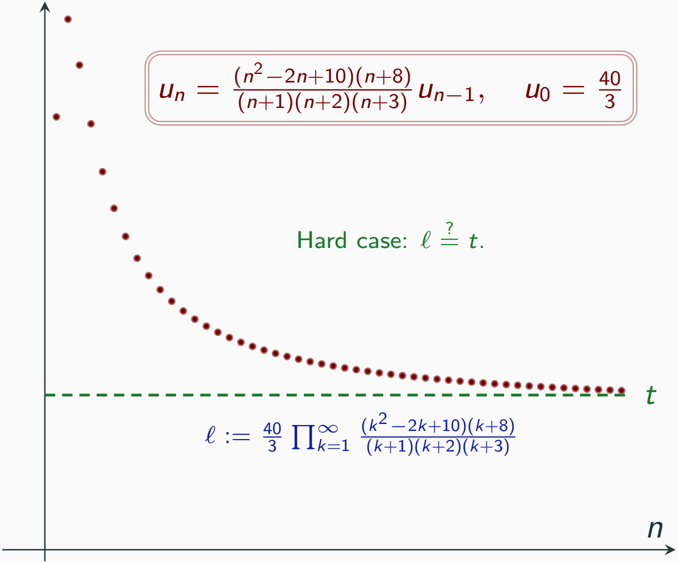

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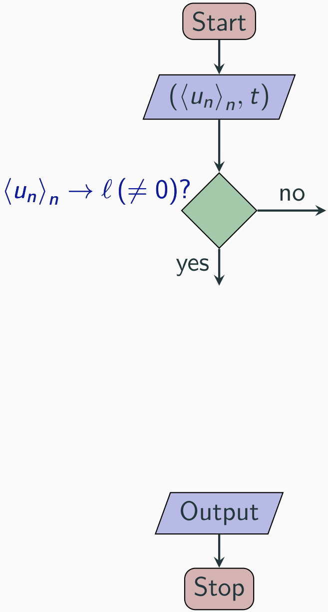

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Finite Search: $u_n \stackrel{?}{=} t$
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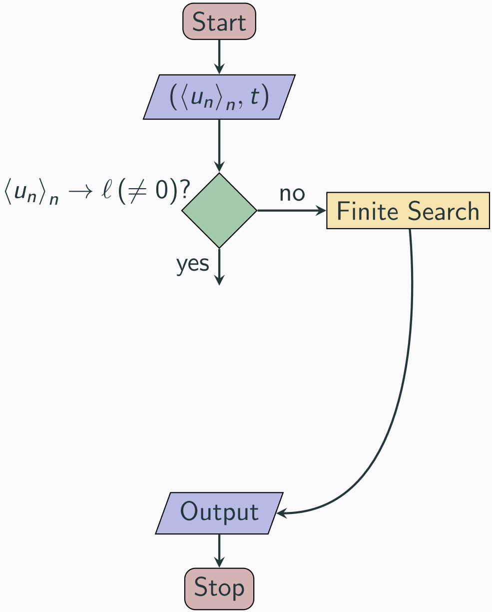

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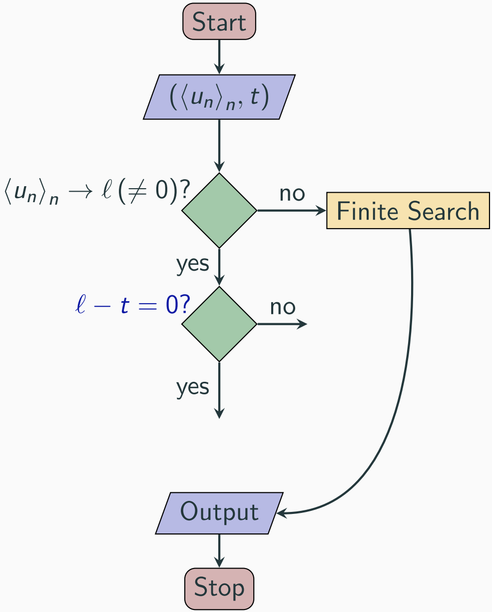

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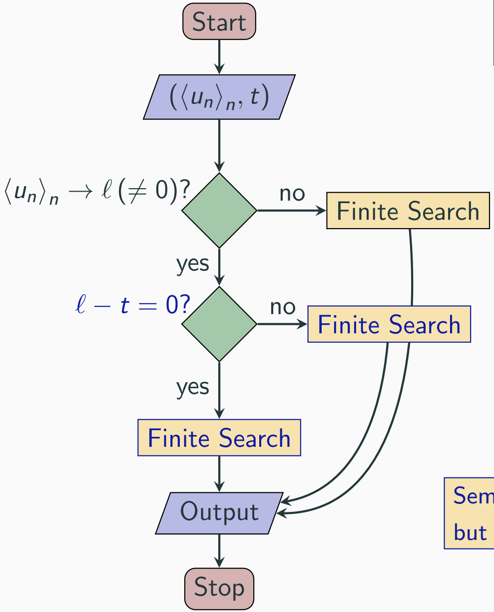

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Semi-algorithm can detect $\ell \neq t$,
but does not terminate for $\ell = t$.

Decidability results for MP in the literature

For hypergeometric sequences $f(n)u_n = g(n)u_{n-1}$ where $f, g \in \mathbb{Z}[x]$ and f and $g\ldots$

- have rational roots, MP is decidable.[1]
- are monic and have roots in a quadratic field, MP is decidable.[2]
- are monic and have quadratic roots MP is (cond.) decidable.[3]
- have quadratic roots there are non-effective results towards MP.[4]

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The p-adic approach to MP circumvents the problematic zero recognition test

Theorem (K, Nosan, Shirmohammadi, and Worrell [ISSAC'23])

For hypergeometric sequences $f(n)u_n = g(n)u_{n-1}$ where $f, g \in \mathbb{Z}[x]$ are monic polynomials such that fg has roots in a quadratic number field, the MP is decidable.

Approach:

Given a sequence $\langle u_n \rangle_n$ and target $t \in \mathbb{Q}$, exhibit an effective threshold B s.t. if $n > B$ then $\exists p$ a prime such that $p \mid u_n$, but $p \nmid t$.

Essentially, we reduce the MP to a finite search problem:
Is $t \in \{u_0, \ldots, u_B\}$?

Here *divisors* is in the p-adic sense (generalising divisors from $\mathbb{Z}$ to $\mathbb{Q}$).

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The p-adic valuation $\nu_p(\cdot)$

For $x \in \mathbb{Z} \setminus \{0\}$,

$$\nu_p(x) = \max\{k \in \mathbb{N}_0 : p^k \mid x\}.$$

For $x \in \mathbb{Q} \setminus \{0\}$,

$$\nu_p(a/b) = \nu_p(a) - \nu_p(b).$$

Examples: $\nu_2(\frac{9}{8}) = -3$ and $\nu_3(\frac{9}{8}) = 2$ since $\frac{9}{8} = 3^2 \cdot 2^{-3}$.

We can also extend the definition to algebraic numbers.

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Essentially, we reduce the MP to a finite search problem:
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We rewrite in terms of the valuations of the poly roots.

$$
\begin{aligned}
\nu_p(u_n) &= \nu_p\left(u_0 \prod_{k=1}^{n} \frac{g(k)}{f(k)}\right), \\
&= \nu_p(u_0) + \sum_{k=1}^{n} \nu_p(g(k)) - \sum_{k=1}^{n} \nu_p(f(k)), \\
&= \nu_p(u_0) + \sum_{k=1}^{n} \sum_{g(\alpha)=0} \nu_p(k-\alpha) - \sum_{k=1}^{n} \sum_{f(\beta)=0} \nu_p(k-\beta).
\end{aligned}
$$

Have we reached the limit of the p-adic approach to MP?

p-adic methods rely on arithmetic results for the distribution of the prime divisors of polynomial products of the form

$$\prod_{k=0}^{n} q(k)$$

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References i

[Ken+23] G. Kenison, K. Nosan, M. Shirmohammadi, and J. Worrell. **"The Membership Problem for Hypergeometric Sequences with Quadratic Parameters".** In: *Proceedings of the 2023 International Symposium on Symbolic and Algebraic Computation*. 2023, pp. 407–416.

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A Transcendental Approach: for some seqs, we can (un)conditionally decide $\ell \stackrel{?}{=} t$ [ICALP'23]. Proof by example:

$$u_n = \frac{(n^2-2n+10)(n+8)}{(n+1)(n+2)(n+3)} u_{n-1}, \quad u_0 = \frac{40}{3}$$

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Theorem (Infinite product evaluation[5])

Suppose that $\sum_{j=1}^{m} \alpha_j = \sum_{j=1}^{m} \beta_j$, *then*

$$\prod_{k=0}^{\infty} \frac{(k+\alpha_1)\cdots(k+\alpha_m)}{(k+\beta_1)\cdots(k+\beta_m)} = \prod_{j=1}^{m} \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}.$$

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Properties of the gamma function

$\Gamma(n) = (n-1)!$, $\Gamma(z+1) = z\Gamma(z)$, and $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$.

Monic and quadratic assumptions permit use of gamma relations of the form:

$$\Gamma(-1-3\mathrm{i})\Gamma(-1+3\mathrm{i}) = \frac{\pi}{30\sinh(3\pi)} \quad \text{and} \quad \frac{\Gamma(1)\Gamma(2)\Gamma(3)}{\Gamma(8)} = \frac{1}{2520}.$$

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$\implies \exists$ non-trivial $P \in \mathbb{Q}[x, y]$ st $P(\mathrm{e}^{\pi}, \pi) = 0$. $\lightning$: e^{π} and π are algebraically independent (Nesterenko, 1996). So $\ell \neq t$. $\square$