A **Forward Construction** of **Inductive Invariants** for **Vector Addition Systems**

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SAMSA Workshop, Warsaw, 04/06/2025

Introduction: inductive invariants

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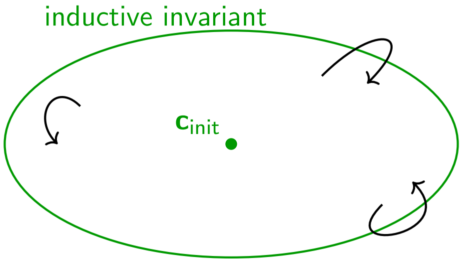

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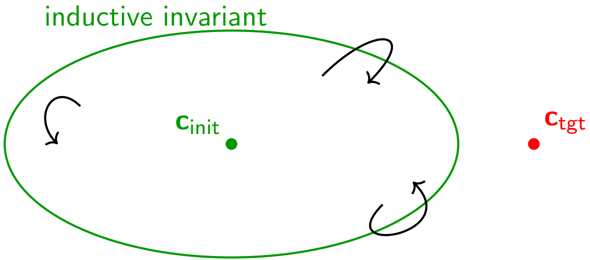

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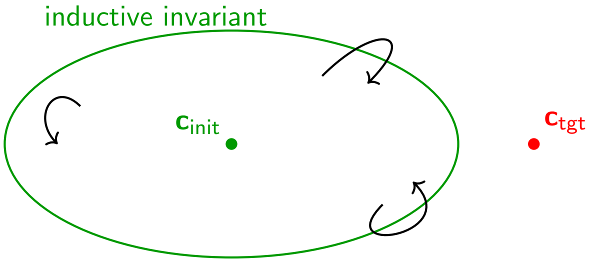

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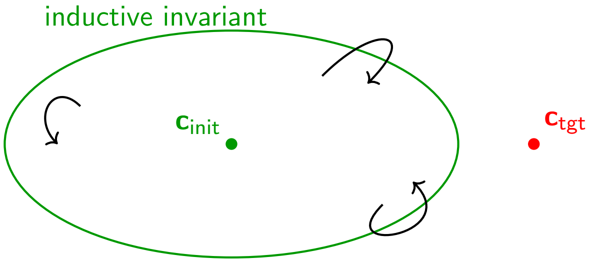

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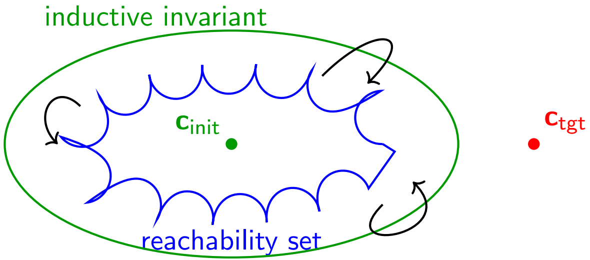

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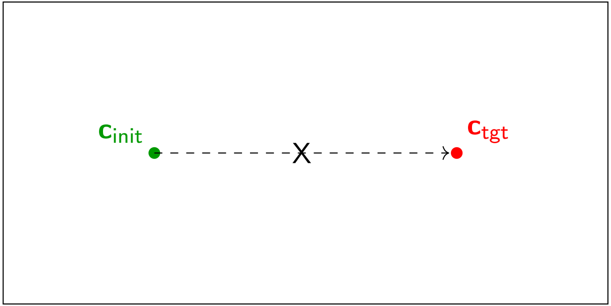
- In 2011, Leroux proved that VAS non-reachability is certified by semilinear sets.

Back-and-Forth Construction of Inductive Invariants



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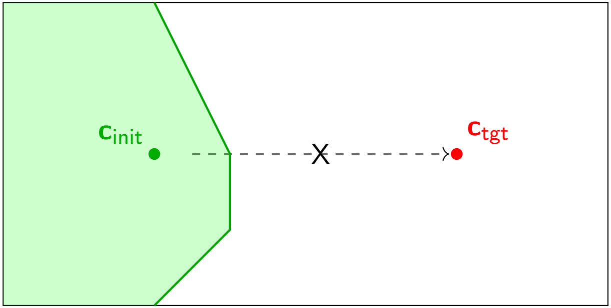

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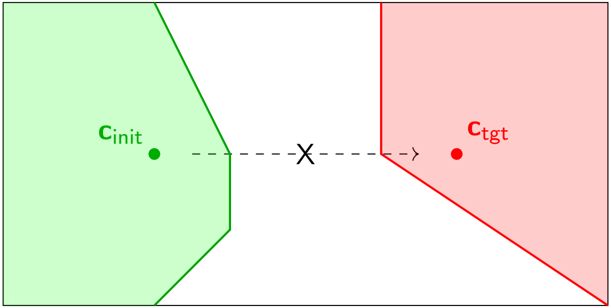

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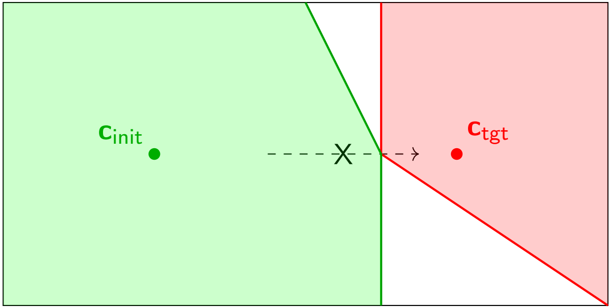

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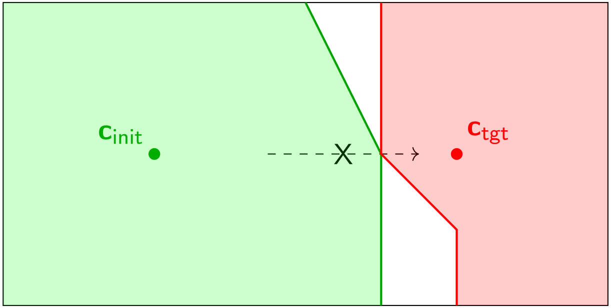

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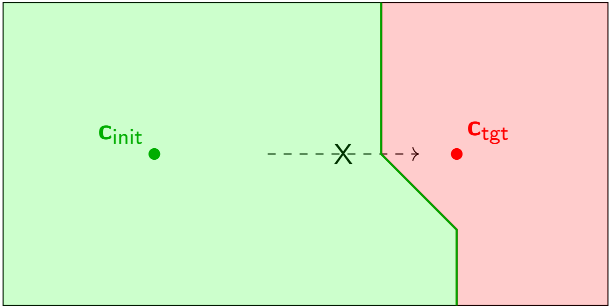

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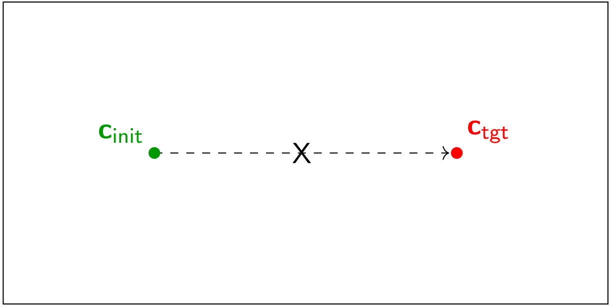

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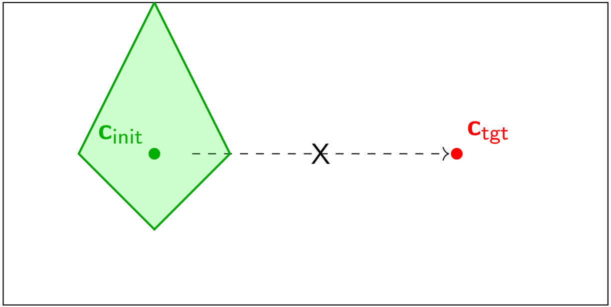

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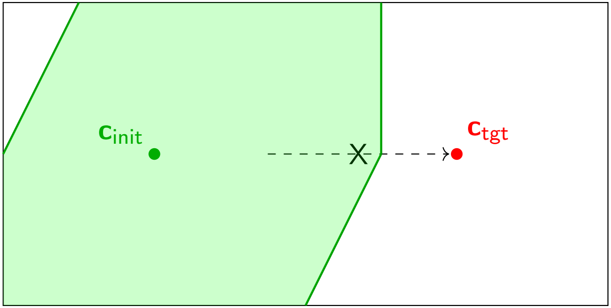

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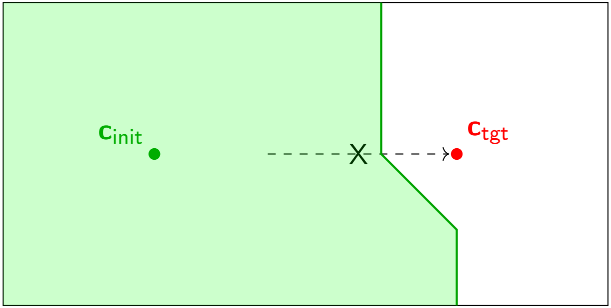

Definitions (VAS + semilinear sets)

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- A semilinear set is a finite union of sets of the form

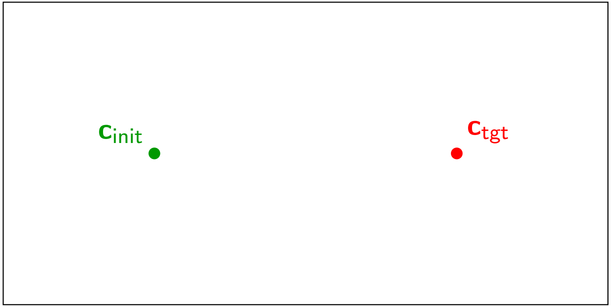
$$\mathbf{b} + \mathbf{P}^* \quad \text{(called linear sets)}$$

for some $\mathbf{b} \in \mathbb{N}^d$ (the basis) and finite $\mathbf{P} \subseteq \mathbb{N}^d$ (the periods), where

$$\mathbf{P}^* := \{\mathbf{p_1} + \ldots + \mathbf{p_n} \mid n \in \mathbb{N}, \mathbf{p_1}, \ldots, \mathbf{p_n} \in \mathbf{P}\}$$

Leroux's Back-and-Forth Construction (in more detail)

Linearization: a tight over-approximation of a VAS reachability set by a semilinear set



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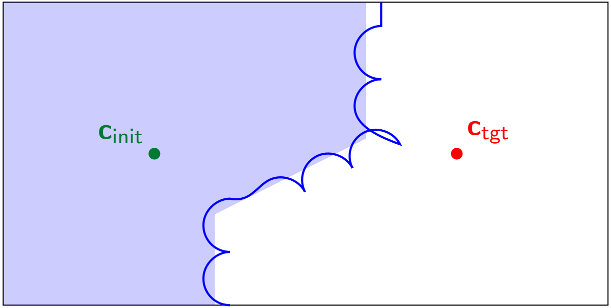

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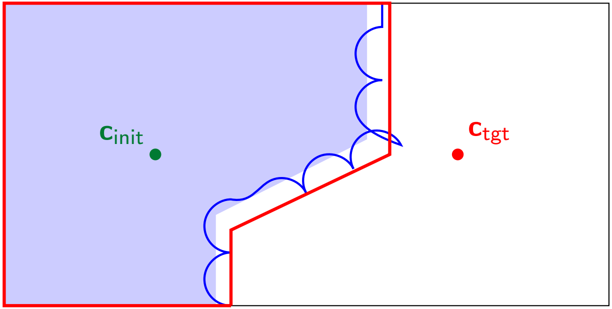

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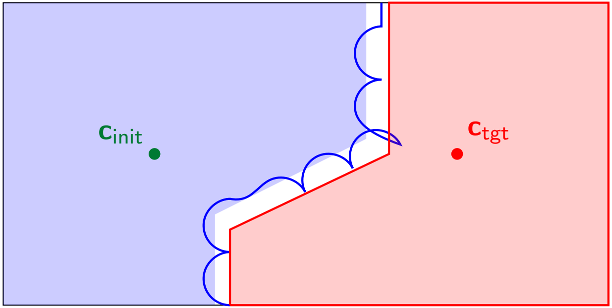

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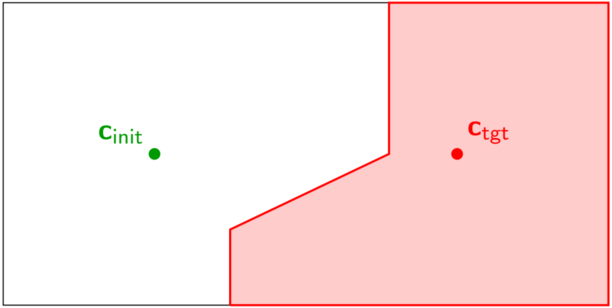

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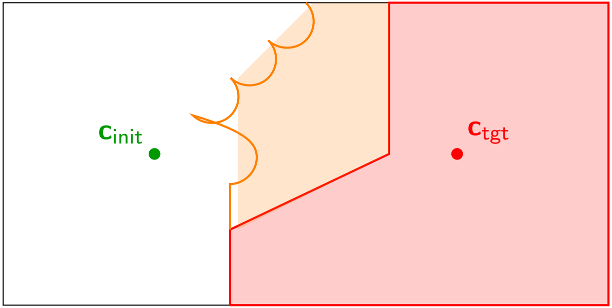

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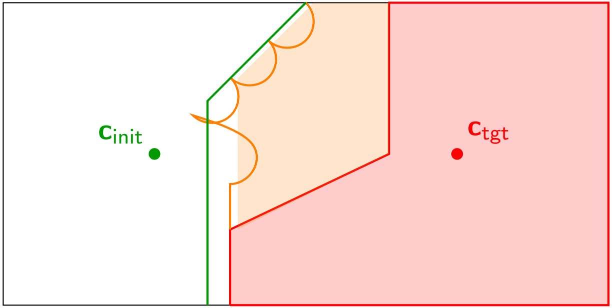

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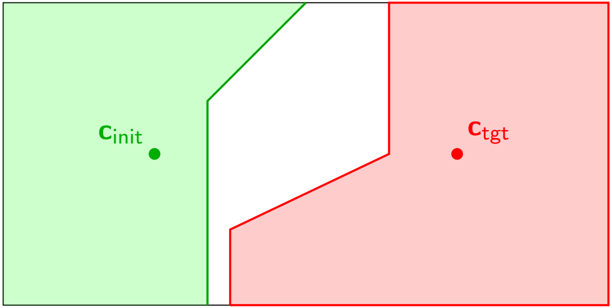

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Goal: Reachability set = finite union of $\mathbf{b} + \mathbf{P}$ where $\mathbf{b} \in \mathbb{N}^d$ and $\mathbf{P} \subseteq \mathbb{N}^d$ is a periodic set (stable by addition + contains 0).

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$$\rho : \quad \mathbf{c}_{\text{init}} \xrightarrow{a_1} \xrightarrow{a_2} \text{end}(\rho)$$

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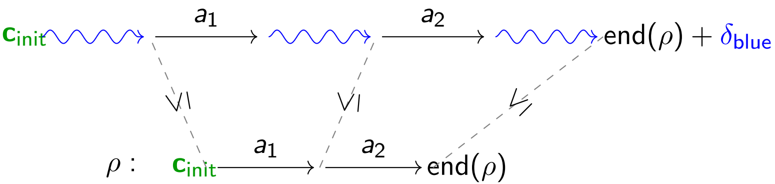

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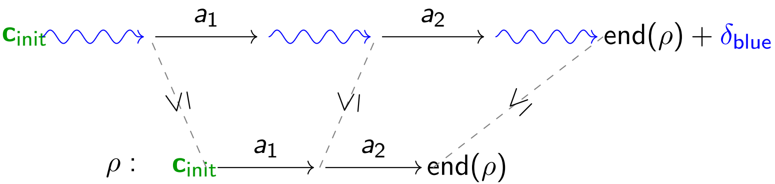

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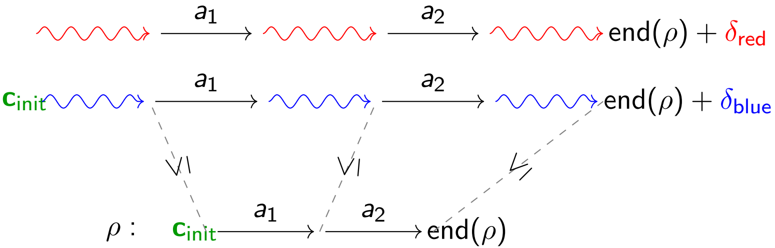

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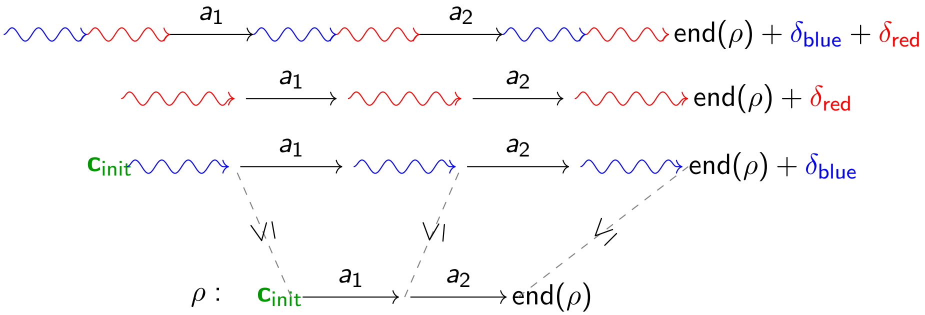

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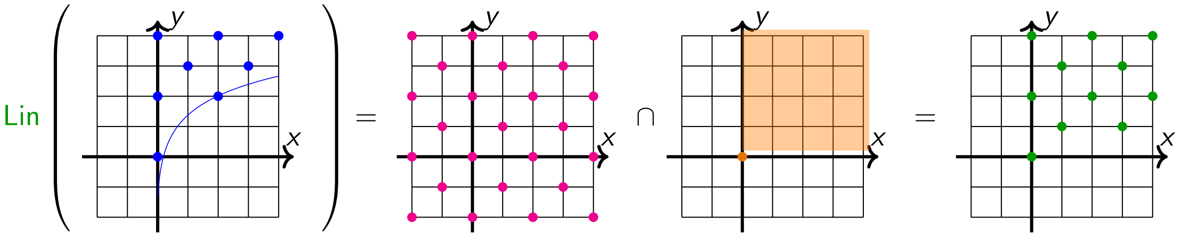

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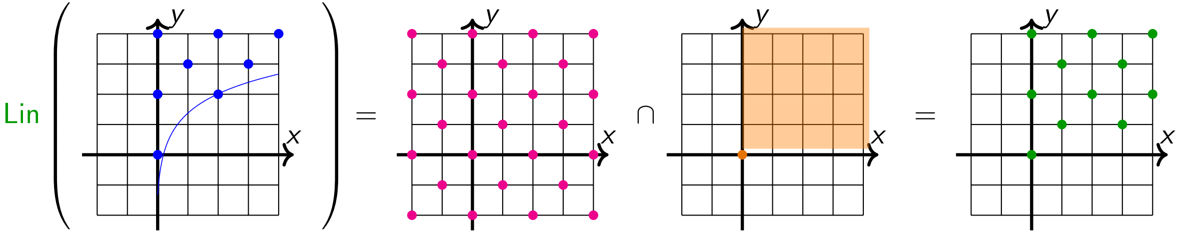

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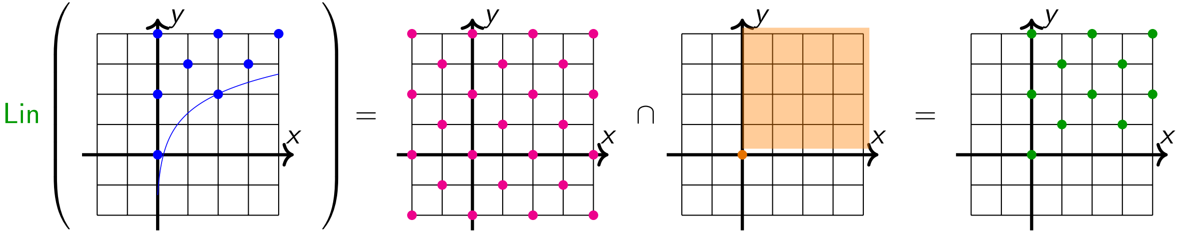

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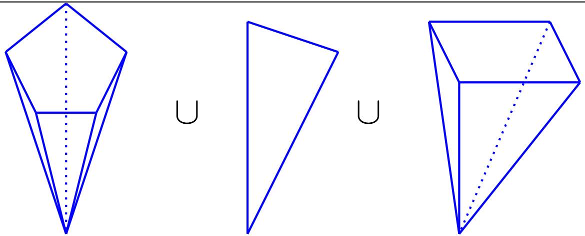
$\{\mathbf{c}_{\text{init}}\}$	

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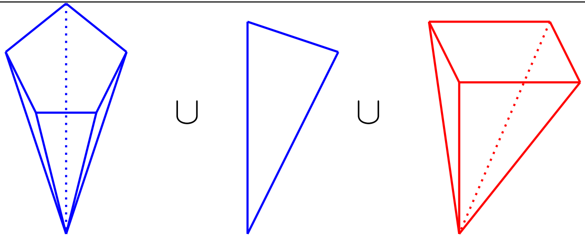
$\{\mathbf{c}_{\text{init}}\}$	

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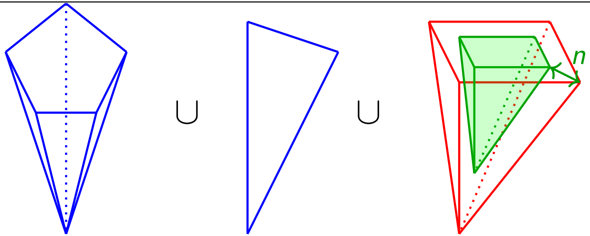
$\{\mathbf{c}_{\text{init}}\}$	

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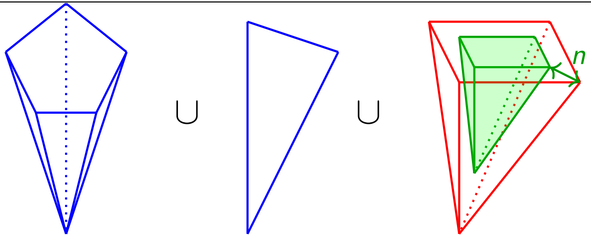
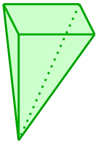
$\{\mathbf{c}_{\text{init}}\}$	
$\{c_{\text{init}}\} \cup$ 	

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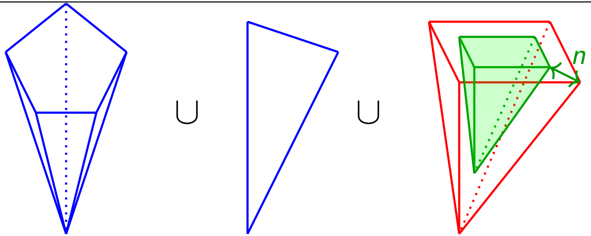
$\{\mathbf{c}_{\text{init}}\}$	
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